Iterated forcing
Part 1: CS iteration

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Outline

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Notation

A forcing notion $(Q, \leq_Q, 1_Q)$ is a preorder with a largest element. (Usually not separative.)
$q \leq p$: q is stronger than p, "has more information than p".
(Note: stronger conditions are always alphabetically later.)

$q \leq^* p$ means:

- $\forall r \leq q$: r is compatible with q.
- Equivalently: For all generic filters G: $q \in G \Rightarrow p \in G$.
- Rephrased: $q \Vdash_Q \check{p} \in G_Q$.

Note: If $q \in Q$, and p is a Q-name of a condition in Q, then $q \leq p$ is still well-defined. *Surprisingly, this is useful!*
We write $q =^* p$ for $q \leq^* p$ and $p \leq^* q$.

Composition and Iteration

Forcing notions:

- $P_0 := \{1\}$.
- $P_1 := Q_0$. (Really: $P_1 = \{1\} * Q_0$).
- $P_2 := Q_0 * \underset{\sim}{Q}_1$ (where $\underset{\sim}{Q}_1$ is a Q_0-name)
  $= \{(p, \underset{\sim}{q}) : p \in Q_0,\ p \Vdash_{Q_0} \underset{\sim}{q} \in \underset{\sim}{Q}_1\}$
- $P_3 := Q_0 * \underset{\sim}{Q}_1 * \underset{\sim}{Q}_2$ ($\underset{\sim}{Q}_2 =$ a $Q_0 * \underset{\sim}{Q}_1$-name)
  $= \{(p_0, \underset{\sim}{p}_1, \underset{\sim}{p}_2) : \cdots\}$

We identify P_n with a subset of P_{n+1} via

$$(q_0, \ldots, q_{n-1}) \mapsto (q_0, \ldots, q_{n-1}, 1_{Q_n}).$$

So: $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$.

Composition, continued

P_n = initital segment, $\underset{\sim}{Q}_n$ = n-th factor

- $P_1 := Q_0$
- $P_2 := Q_0 * \underset{\sim}{Q}_1$
- $P_3 := Q_0 * \underset{\sim}{Q}_1 * \underset{\sim}{Q}_2$

Universes:

- $V_0 := V[G_0] = V$, where $G_0 \subseteq P_0$ is trivial.
- $V_1 := V[G_1]$, $G_1 \subseteq P_1$ is V-generic
- $V_2 := V[G_2]$, $G_2 \subseteq P_2 = Q_0 * \underset{\sim}{Q}_1$ is V-generic.

or: $V_2 := V[G(0)][G(1)] = V[G(0) * G(1)$, where

- $G(0) = G_{Q_0} \subseteq Q_0$ is V-generic,
- $G(1) = G_{Q_1} \subseteq Q_1[G(0)]$ is $V[Q_0]$-generic.

Definition

A forcing iteration of length γ is a sequence $(P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \gamma)$ (often together with P_γ) where:

- P_α is a set of partial functions with domain $\subseteq \alpha$
- $\leq_{P_\alpha}$ is preorder on P_α, defined recursively, see below. Thus P_α becomes a forcing notion (with largest element $\emptyset$).
- For all α, $\underset{\sim}{Q}_\alpha$ is a P_α-name of a forcing notion
- For all $\alpha_1 < \alpha_2 \leq \gamma$: $P_{\alpha_1} \subseteq P_{\alpha_2}$.
- For $p', p \in P_\alpha$: $p' \leq_{P_\alpha} p$ iff for all $\beta < \alpha$: $p' \restriction \beta \Vdash_{P_\beta} p'(\beta) \leq_{Q_\alpha} p(\beta)$. (If $p'(\beta)$ and/or $p(\beta)$ are undefined, replace them by 1_{Q_α}.)
- For all $\alpha < \gamma$: $P_{\alpha+1} = \{p : p\restriction\alpha \in P_\alpha,\ p\restriction\alpha \Vdash_{P_\alpha} p(\alpha) \in \underset{\sim}{Q}_\alpha\}$, so $P_{\alpha+1} = P_\alpha * \underset{\sim}{Q}_\alpha$ (up to a natural isomorphism).

Definition

A forcing iteration of length γ is a sequence $(P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \gamma)$ together with P_γ where ...

Example

An iteration of length 2 is of the form $(P_0, Q_0, P_1, Q_1, P_2)$, where P_2 is "the same" as $Q_0 * Q_1$.

Notation

We will write G_{P_α} or G_α for V-generic filters on P_α, and G_{Q_α} or $G(\alpha)$ for $V[G_\alpha]$-generic filters on $\underset{\sim}{Q}_\alpha[G_\alpha]$.

Limits, quotients

If $(P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta)$ is an iteration with P_δ undefined (a "topless" iteration), there are several ways to defined P_δ consistently (i.e., such that the definition of iteration is still satisfied):

- $P_\delta := \bigcup_{\alpha<\delta} P_\alpha$, the "direct limit"
- $P_\delta := \{p \mid \forall\alpha : p{\restriction}\alpha \in P_\alpha\}$, the "projective limit" or "full limit"
- others ...

Definition

If $(P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta)$ is an iteration with (some) limit P_δ, then for all $\alpha \le \delta$ we define a P_α-name P_δ/P_α (also called P_δ/G_α) by

$$P_\delta/P_\alpha := \{p \in P_\delta : p{\restriction}\alpha \in G_\alpha\}$$

$$\Vdash_{P_\alpha} P_\delta/P_\alpha := \{p \in P_\delta : p{\restriction}\alpha \in G_\alpha\}$$

This is a P_α-name of a forcing notion. (order: inherited)
Note that "$p{\restriction}\alpha$ carries no information (except for ensuring that $p{\restriction}[\alpha,\delta)$ is in the quotient)", because they are all compatible. In other words, if $p{\restriction}[\alpha,\delta) = p'{\restriction}[\alpha,\delta)$, and both $p{\restriction}\alpha$, $p'{\restriction}\alpha$ are in G_α, then in P_δ/P_α we have $p =^* p'$.
So an alternative definition would be
$P_\delta/P_\alpha := \{p{\restriction}[\alpha,\delta) : p \in P_\delta : p{\restriction}\alpha \in G_\alpha\}$.

Fact

- *P_δ/P_α is (morally) an iteration of length $\delta - \alpha$.*
- *P_δ is canonically (densely) embedded into $P_\alpha * (P_\delta/P_\alpha)$.*

(If δ is additively indecomposable, then the remainder iteration will be again of length δ. This allows "bookkeeping arguments".)

Outline

Let Q be a forcing notion.

Definition

Q is proper iff:

- for every sufficiently closed countable set N containing Q (usually: N is a countable elementary submodel of a large initial segment of the universe),
- and for all $p \in Q \cap N$

there is $q \leq p$ which is N-generic.

Definition

"q is N-generic" means that

For all maximal antichains $A \in N$: $q \Vdash A \cap G \subseteq N$.

(Note that for uncountable antichains A, $A \setminus N$ will be nonempty. q forces that these sets are avoided by G.)

Properness rephrased

"Q is proper" means:

- **for all** countable $N \prec V$ (with $P \in N$), all $p \in Q \cap N$
- **there is** $q \leq p$ (typically $q \notin N$) such that:

  for all names $\underset{\sim}{\alpha}$ of ordinals with $\underset{\sim}{\alpha} \in N$
  we have $q \Vdash \underset{\sim}{\alpha} \in N$.
  (I.e., for all generic G containing q: $\underset{\sim}{\alpha}[G] \in N$.)

We have: ccc $\Rightarrow$ proper $\Rightarrow$ ω_1^V stays uncountable.
All forcing notions considered in this talk will satisfy the $\aleph_2$-cc (using CH and the Δ-lemma), so no cardinals will be collapsed.

Properness via games

For any forcing notion Q and all $p \in Q$, the game $G(Q, p)$ is defined as follows:

- There are ω many innings (rounds).
- In the k-th inning, player I (=bad player) first plays a name $\underaccent{\sim}{A}_k$ of a countable set of ordinals.
  Player II (=good player) then responds with a countable set B_k of ordinals.
- At the end, player II wins if there is a condition $q \leq p$ forcing $\bigcup_k \underaccent{\sim}{A}_k \subseteq B_k$.

Lemma

Q is proper iff player II has a winning strategy for each $G(Q, p)$.

Equivalent variants: Either/Both players play only singletons.

Properness game: Player I plays a name $\underset{\sim}{\alpha}_k$ of an ordinal, player II responds with a countable set B_k of ordinals. Player II wins if at the end there is a condition q forcing

$$\forall k : \underset{\sim}{\alpha}_k \in \bigcup_n B_n$$

Proper$^+$ness, a strong variant of properness (related to $(\omega, 1)$-properness)

- In the k-th inning, player I plays a name $\underset{\sim}{\alpha}_k$ of an ordinal Player II responds with a countable set B_k of ordinals.
- At the end, player II wins if there is a condition $q \leq p$ forcing $\forall k : \underset{\sim}{\alpha}_k \in B_k$.

Even stronger (because it implies ω^ω-bounding): each B_k is finite.

Outline

An iteration $(P_\alpha, Q_\alpha : \alpha < \delta)$ is called a countable support iteration (CS iteration) iff:

- For each limit $\varepsilon < \delta$ of cofinality ω, P_ε is the full limit of $(P_\alpha, Q_\alpha : \alpha < \varepsilon)$.
- For each limit $\varepsilon < \delta$ of uncountable cofinality, P_ε is the direct limit of $(P_\alpha, Q_\alpha : \alpha < \varepsilon)$.

For any such iteration we define its countable support limit P_δ in the obvious way (depending on $cf(\delta)$, as a direct or full limit).
Equivalently: P_δ is the set of all partial functions p with countable domain such that for all α, $p{\restriction}\alpha \Vdash p(\alpha) \in Q_\alpha$.
For simplicity, in this talk we consider iterations $(P_n, Q_n : n < \omega)$ (with CS limit P_ω) only.

Iteration theorem for CSI

Theorem (Properness preservation, weak version)

Let $(P_n, \underset{\sim}{Q}_n : n < \omega)$ be a CS iteration with CS limit P_ω, and assume that $\Vdash_{P_n}$ "$\underset{\sim}{Q}_n$ is proper". Then P_ω is proper.
($\forall N\ \forall p \in P_\omega \cap N\ \ \exists q \le p$, (P_ω, N)-generic.)

Theorem (Properness preservation, strong version)

Let P_n, Q_n, P_ω, N be as above. Assume that $n_0 < \omega$.
Let $\underset{\sim}{p}$ be a P_{n_0}-name. Assume that $q_{n_0} \in P_{n_0}$ is (P_{n_0}, N)-generic, and $q_{n_0} \le^* \underset{\sim}{p}{\restriction}n_0$. (I.e., $q_{n_0} \Vdash \underset{\sim}{p}{\restriction}n_0 \in G_{n_0}$.)
Then for all k with $n_0 \le k \le \omega$ there is a $q_k \le^* \underset{\sim}{p}{\restriction}k$ such that:

1. *$q_k{\restriction}n_0 = q_0$. (Moreover $\forall i < j$: $q_i = q_j{\restriction}i$)*
2. *q_k is (P_k, N)-generic.*

Easy to prove for successor steps $k \mapsto k + 1$.

Properness of P_ω

Given: $(P_n, Q_n : n \in \omega)$, $N \prec V$, $p \in P_\omega \cap V$, $q_0 \leq p{\restriction}n_0$, q_0 is $(N, P_{n_0}$-generic.
Want: $q \in P_\omega$, $q \leq p$, N-generic, q continues q_0.
For notational simplicity let $n_0 := 0$.

- **Strategy 1** Extend $q_0 \in P_0$ to $q_1 \in P_1$, still generic, and below $p{\restriction}(1)$. Then to q_2, etc.
- **Strategy 2** Start with an enumeration $(\underset{\sim}{\alpha}_k : k \in \omega)$ of all ordinal names in N. In step k, make sure that q_k forces α_k into N.
- **Strategy 3** (blackboard, next slide)

Properness of P_ω, continued

GIVEN: $p_0 \in P_\omega \cap V$. WANT: $q \in P_\omega$, $q \leq p$, N-generic.

Proof.

Start with an enumeration $(\underset{\sim}{\alpha}_k : k \in \omega)$ of all ordinal names in N.
Find in N a Q_0-name of a condition $\underset{\sim}{p}_1 \in P_\omega/P_1$ such that the following is forced:

> *If $p{\restriction}1 \in G_1$, then $p_1 \leq p_0$.*
> *Moreover, p_1 decides (in P_ω/P_1) the value of $\underset{\sim}{\alpha}_1$ as $\underset{\sim}{\beta}_1$.*
> *(Recall: P_ω-name, P_1-name.)*

Now find $q_1 \in P_1$, generic, stronger than $p_0{\restriction}1$.
q forces: $\underset{\sim}{\beta}_1 \in N$, $p_1 \leq p_0$, $p_1 \in P_\omega/P_1$.
Continue by first finding $\underset{\sim}{p}_2 \in P_\omega/P_2$, then q_2 extending q_1.
Check that $q_\omega = \bigcup_n q_n$ is P_ω-generic.

Properness preservation using games

Start with a condition p.
Player I plays a P_ω-name $\underset{\sim}{\alpha}_0$, player II plays a condition $p_0 \leq p$ forcing $\underset{\sim}{\alpha}_0 = \beta_{00}$.
Player I plays a P_ω-name $\underset{\sim}{\alpha}_1$, player II plays (in V^{P_1}, assuming $p_0 \restriction 1 \in G_1$) a condition $p_1 \in P_\omega / P_1$, $\underset{\sim}{p}_1 \leq p_0$, forcing $\alpha_1 = \underset{\sim}{\beta}_{11}$. NOTE: $\underset{\sim}{p}_1$, $\underset{\sim}{\beta}_{11}$ are P_1-names!)
Moreover, player II starts the game $G(Q_0, p_0(0))$ and replies to the Q_0-name β_{11} with an ordinal β_{10}.
Player I plays a P_ω-name $\underset{\sim}{\alpha}_2$, player II plays (in V^{P_2}, assuming $p_1 \restriction 2 \in G_2$) a condition $\underset{\sim}{p}_2 \in P_\omega / P_2$, $\underset{\sim}{p}_2 \leq \underset{\sim}{p}_1$, forcing $\alpha_1 = \underset{\sim}{\beta}_{22}$.
Moreover, player II starts (in V^{P_1}) the game $G(Q_1, p_2(2))$ and replies to the $Q_0 * Q_1$-name β_{22} with a Q_0-name $\underset{\sim}{\beta}_{21}$. Finally, replies in $G(Q_0, p_1(1))$ to $\underset{\sim}{\beta}_{21}$ with an ordinal β_{20}.
Etc. Winning all games yields a generic condition.

WARNING:

Sometimes theorems are important.
Sometimes proofs are important.

This proof is important, because many later proofs of stronger theorems are just more sophisticated versions of this proof.

(Compare: proof of forcing theorem.)

Preservation theorems

Let P_ω be the CS-limit of $(P_n, Q_n : n < \omega)$.

Theorem (weak version, P/P)

If all P_n are nice, then P_ω is nice.

Theorem (strong version, Q/P)

IF $\forall n : \Vdash_{P_n}$"Q_n is nice", THEN P_ω is nice.

Note: Necessary for this to work: the condition

$$\textit{If } Q_0 \textit{ is nice and} \Vdash_{Q_0} \underset{\sim}{Q}_1 \textit{ is nice, then } Q_0 * \underset{\sim}{Q}_1 \textit{ is nice}$$

should be trivial. (Or at least: true.)
These theorems are true for many versions of "nice", such as: "ω^ω-bounding", or "Laver property".