Iterated forcing, Part 3: FS iteration and small forcings

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Outline

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Sorry. . .

A correction concerning yesterday's talk.
I claimed that a V-generic filter $G \subseteq \prod_i Q_i$ will induce filters $G(i) \subseteq Q_i$ . . .

- . . . which are V-generic subsets of Q_i.
  (That is true!)
- . . . which are not $V[G(j)]$-generic.
  (That was false, as several of you have pointed out.)

What I should have said: Assume $i \neq j$.

- The forcing notions Q_i are given by definitions in V. ("Set of all sequences . . . ")
- The same definition will give a forcing notion Q_i' in $V[G(j)]$.
- $V[G(j)] \models Q_i \neq Q_i'$ in general, and often not even $Q_i < Q_i'$.
- The filter $G(i)$ will be Q_i-generic over $V[G(j)]$, but in general not Q_i'-generic.

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Definition

An iteration $(P_\alpha, Q_\alpha : \alpha < \delta)$ is called a FS iteration iff:

- For each limit $\varepsilon < \delta$ of cofinality ω, P_ε is the direct limit of $(P_\alpha, Q_\alpha : \alpha < \varepsilon)$ (i.e., $P_\varepsilon = \bigcup_{\alpha<\varepsilon} P_\alpha$).
- Equivalently: Each P_β is the set of all partial functions p with finite domain $\subseteq \beta$, s.t. for all α: $p{\restriction}\alpha \Vdash p(\alpha) \in Q_\alpha$.

For any such (topless) iteration we define its finite support limit P_δ as the direct limit. We write $\Vdash_\alpha$ instead of $\Vdash_{P_\alpha}$.

Theorem

1. *If for all $\alpha < \delta$ we have $\Vdash_\alpha Q_\alpha \models$ ccc, then also $P_\delta \models$ ccc.*
2. *If for all $n < \omega$ we have $Q_n \not\models$ ccc, then $\Vdash_\omega \omega \approx \omega_1^V$.* *("$P_\omega$ collapses $\aleph_1$.")*

From now on we only consider FS iterations of ccc forcings. We may want to start with $\neg$CH.

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A fragment of Cichoń's Diagram

$$
\begin{array}{ccccccc}
 & & & & & & \mathfrak{c} \\
 & & & & & & \uparrow \\
\mathrm{cov}(\mathcal{N}) & \rightarrow & \mathrm{non}(\mathcal{M}) & \rightarrow & \cdots & \rightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \rightarrow & \cdots & & \\
 & & \uparrow & & \uparrow & & \\
\mathrm{add}(\mathcal{N}) & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & \cdots \\
\uparrow & & & & & & \\
\omega_1 & & & & & &
\end{array}
$$

- How many $\mathcal{N}$ sets (=sets of Lebesgue measure 0, null sets) do we have to add together (in the sense of $\cup$) to get a non-null set?
- How many null sets do we need to cover the real line?
- How many points do we need to get a non-$\mathcal{M}$eager set?
- ...

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Increasing cov($\mathcal{N}$) and $\mathfrak{b}$

Definition

We write $\mathbb{B}$ for random forcing. $\mathbb{B}$ adds a real that avoids every Borel measure zero set whose code is in the ground model.
We write $\mathbb{D}$ for Hechler forcing. $\mathbb{D}$ adds a function in ω^ω which dominates all old functions.

Fact

Let λ be regular uncountable.
Let $(P_\alpha, Q_\alpha : \alpha < \lambda)$ be an iteration where cofinally often we have $Q_\alpha = \mathbb{B} =$ random forcing.
Then $\Vdash_\lambda$ cov($\mathcal{N}$) $\geq \lambda$.

Proof.

Every small family of null sets appears in an intermediate model (use ccc!), so the next random real is not covered.

Fact

Replacing $\mathbb{B}$ by $\mathbb{D}$ we get a model where $\mathfrak{b} \geq \lambda$. (Every small set is bounded.)

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Increasing $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$, a bit

Definition

Let κ be a cardinal, P a forcing notion. A κ-subforcing of P is a nice subset of P of size κ, typically $P \cap N$ for some elementary model, or $P \cap V_0$ for some "earlier" model V_0 in an iteration.

Subforcings must agree on $\leq$ and $\perp$, so subforcings of ccc forcings are again ccc.

Fact

- *Let λ be regular uncountable, $\kappa_{cn} \leq \lambda$. Let $(P_\alpha, Q_\alpha : \alpha < \lambda)$ be an iteration where every $< \kappa_{cn}$-sized subforcing of $\mathbb{B}$ appears somewhere as Q_α. Then $\Vdash_\lambda \mathrm{cov}(\mathcal{N}) \geq \kappa_{cn}$.*
- *Similarly: Let $\kappa_b \leq \lambda$. Let $(P_\alpha, Q_\alpha : \alpha < \lambda)$ be an iteration where every $< \kappa_b$-sized subforcing of $\mathbb{D}$ appears somewhere as Q_α. — Then $\Vdash_\lambda \mathfrak{b} \geq \kappa_b$.*

Combining these two constructions yields a model of $\mathrm{cov}(\mathcal{N}) = \kappa_{cn} \leq \kappa_b = \mathfrak{b}$.

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How did we keep $\mathfrak{b}$ small?

Definition

A sequence $\vec{f} = (f_i : i < \kappa)$ is a scale in $(\omega^\omega, \leq^*)$ if

- For all $i < j$ we have $f_i \leq^* f_j$.
- $\vec{f}$ is unbounded, i.e.: there is no g with $\forall i : f_i \leq^* g$.

(Note: $\vec{f}$ is not necessarily dominating.)

Recall that $\mathfrak{b}$ is the shortest length of a scale.

Theorem (How to keep $\mathfrak{b} \leq \kappa$)

Let $\vec{f} = (f_i : i < \kappa)$ be a scale in $(\omega^\omega, \leq^*)$, κ regular. Then:

1. *If Q is a forcing of size $< \kappa$, then $\Vdash_Q$ "$\vec{f}$ is a scale".*

Proof.

Easy. (Blackboard)

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An ambitious plan

Does the following plan work?
To get a model where a specific cardinal $\mathfrak{x}$ has value κ, and the continuum has value λ, try this:

- Find a (nice) forcing notion Q which "increases $\mathfrak{x}$". (Nice = Souslin ccc, i.e.: the relations $\leq_Q$ and also $\perp_Q$ are analytic — often even Borel)
- (For example, if $\mathfrak{x} = \mathrm{cov}(I)$, where I is a Borel ideal, find a forcing such that the generic object is in no set from I which comes from the ground model. Similar to random/Hechler from before.)
- Use an iteration $(P_\alpha, Q_\alpha : \alpha < \lambda)$ where each $< \lambda$-sized subforcing of Q appears as some Q_α.
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To get a model where a specific cardinal $\mathfrak{x}$ has value κ, and the continuum has value λ, try this:

- Find a (nice) forcing notion Q which "increases $\mathfrak{x}$".
  (Nice = Souslin ccc, i.e.: the relations $\leq_Q$ and also $\perp_Q$ are analytic — often even Borel)
- (For example, if $\mathfrak{x} = \mathrm{cov}(I)$, where I is a Borel ideal, find a forcing such that the generic object is in no set from I which comes from the ground model. Similar to random/Hechler from before.)
- Use an iteration $(P_\alpha, Q_\alpha : \alpha < \lambda)$ where each $< \lambda$-sized subforcing of Q appears as some Q_α.
- Hope that $\Vdash_\lambda \mathfrak{x} = \kappa, \mathfrak{c} = \lambda$.

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No, this will often not work.

Examples (of failure)

- There are no "small subforcings of Cohen". Every iteration of length λ (λ regular $\geq \mathfrak{c}$) will force $\mathrm{cov}(\mathcal{M}) = \lambda$.
- Subforcings of nice forcings can be very naughty. For example, there may be a subforcing of $\mathbb{B}$ which adds a dominating real. (Even though $\mathbb{B}$ is ω^ω-bounding.)

Example (of success)

Let $(P_\alpha, Q_\alpha : \alpha < \aleph_{\omega+1})$ be an iteration in which each Q_α is a (cleverly chosen) subforcing of $\mathbb{B}$ of size $< \aleph_\omega$.
Then $\Vdash_{\aleph_{\omega+1}} \mathrm{cov}(\mathcal{N}) = \aleph_\omega$. (!!!)

Remark

- Difficult.
- In contrast, $\mathrm{cov}(\mathcal{M})$ must have cofinality $\geq \mathrm{add}(\mathcal{N}) \geq \omega_1$.

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Outline

Many different cardinals

Theorem (G-Mejía-Shelah)

There is a model in which all the displayed cardinals have different values.

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 & & & & & & \mathfrak{c} \\
 & & & & & & \uparrow \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \rightarrow & (\mathfrak{c}) & & \\
 & & \uparrow & & \uparrow & & \\
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This model can be obtained using the technique of "small" forcings. However, the small forcing notions increasing $\mathrm{non}(\mathcal{M})$ have to be chosen carefully, as they threaten to add dominating reals and hence increase $\mathfrak{b}$.

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Increase non($\mathcal{M}$) without increasing $\mathfrak{b}$

Definition

The forcing notion $\mathbb{E}$ is the set of all conditions $p = (s^p, w^p, \varphi^p)$ where:

- $s \in \omega^{<\omega}$
- $w \in \omega$
- $\varphi = (\varphi_k : k \in \omega)$ is a family of sets in $[\omega]^{\leq w}$ (a “slalom” of bounded width w)
- $\forall i < |s| : s_i \notin \varphi_i$

The generic object g will be a sequence in ω^ω. p forces that g extends s and avoids all sets in φ: $g(i) \notin \varphi_i$. g defines a meager set $M_g = \{x \in \omega^\omega : \forall^\infty i\, x(i) \neq g(i)\}$ (“eventually different”). Every real from the ground model will be in M_g.

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Thank you!

Your patience will be rewarded.