Menger spaces and their relatives: basic facts

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Menger spaces and relatives

A topological space X is *Menger* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

A topological space X is *Hurewicz* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X.

A topological space X is *Scheepers* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a ω-cover of X.

$\mathcal{U}$ is an ω-cover of X if $\forall F \in [X]^{<\omega} \exists U \in \mathcal{U} (F \subset U)$.

$\mathcal{U}$ is a γ-cover of X if $\forall x \in X \forall^* U \in \mathcal{U} (x \in U)$.

σ-compact $\to$ Hurewicz $\to$ Scheepers $\to$ Menger $\to$ Lindelöf.

Example: ω^ω is not Menger. Witness:
$\mathcal{U}_n = \big\{\{x : x(n) = k\} : k \in \omega\big\}$.

Folklore Fact. For analytic sets of reals Menger is equivalent to σ-compact.

In L there exists a co-analytic Menger subspace of ω^ω which is not σ-compact.

Examples under CH.

$X \subset \omega^\omega$ is a *Luzin* set if $|X| = \omega_1$ and $|X \cap M| \leq \omega$ for any meager M. Every Luzin set is Menger because concentrated.

$X \subset 2^\omega$ is a *Sierpinski* set if $|X| = \omega_1$ and $|X \cap N| \leq \omega$ for any measure 0 set N. Every Sierpinski set is Hurewicz because of the following characterization due to Scheepers

Theorem

Let P be compact. $X \subset P$ is Hurewicz iff for every G_δ-set $G \supset X$ there exists a σ-compact F such that $X \subset F \subset G$.

Proof. $(\rightarrow)$. Let $G = \bigcap_{n\in\omega} O_n$. Set $\mathcal{U}_n = \{U : U \subset P$ is open and $\bar{U} \subset O_n\}$. Let $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ be such that $\{\cup\mathcal{V}_n : n \in \omega\}$ is a γ-cover of X. Then $X \subset \bigcup_{n\in\omega} \bigcap_{m\geq n} \cup\mathcal{V}_m \subset G$. □

Corollary

Luzin sets are not Hurewicz.

ZFC examples

Given $x, y \in \omega^\omega$, $x \leq^* y$ means $\{n : x(n) \leq y(n)\}$ is cofinite. $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ω^ω. $\mathfrak{d}$ is the minimal cardinality of an unbounded subset of ω^ω.

$|X| < \mathfrak{b} \to X$ is Hurewicz. $\mathfrak{b}$- Sierpinski sets are Hurewicz.
$|X| < \mathfrak{d} \to X$ is Menger (even Scheepers). $\mathfrak{d}$- Luzin sets are Menger.

A set $X \subset \omega^\omega$ is κ-concentrated on a countable Q, if $|X| \geq \kappa$ and $|X \setminus U| < \kappa$ for any open $U \subset \omega^\omega$ containing Q. If $\kappa \leq \mathfrak{d}$, then $X \cup Q$ is Menger.

Fact. There exists a $\mathfrak{d}$-concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\leq^* d_\beta$ for all $\beta \leq \alpha$. Viewed as a subspace of $(\omega+1)^{\uparrow\omega}$, S is $\mathfrak{d}$-concentrated on $Q = \{x \in (\omega+1)^{\uparrow\omega} : x$ is eventually $\omega\}$. □

Fact. There exists a $\mathfrak{b}$-concentrate set.

Proof. Fix an unbounded $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$ such that $b_\beta \leq^* b_\alpha$ for all $\beta \leq \alpha$. B is $\mathfrak{b}$-concentrated on Q. □

Nontrivial (Bartoszynski-Shelah): $B \cup Q$ is Hurewicz. "All $\mathfrak{b}$-concentrated sets are Hurewicz" is independent.

Preservation by unions

Like all reasonable covering properties, Menger, Scheeprs and Hurewicz ones are preserved by continuous images and closed subspaces. If X is Menger (Scheepers, Hurewicz) and K is compact, then so is $X \times K$.

Fact. Menger and Hurewicz properties are preserved by countable unions. Hence also by products with σ-compacts.

Proof. Let $X = \bigcup_{k\in\omega} X_k$ and $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. Let $\langle \mathcal{V}^k_n : n \in \omega \rangle$ be such that $\mathcal{V}^k_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup\mathcal{V}^k_n : n \in \omega\}$ is a large (resp. γ-)cover of X_k. Set $\mathcal{V}_n = \bigcup_{k\leq n} \mathcal{V}^k_n$. □

Corollary

Menger and Hurewicz properties are preserved by unions of families of size $< \mathfrak{b}$. □

Proposition

$\mathrm{add}(\mathrm{Menger}) \in [\min\{\mathfrak{b}, \mathfrak{g}\}, \mathrm{cf}(\mathfrak{d})]$ □.

Preservation by products

Fact. (CH.) There are two Sierpinski (hence Hurewicz) sets S_0, S_1 whose product is not Menger.

Proof. Fix a countable dense $Q \subset 2^\omega$ and write $2^\omega \setminus Q = \{x_\alpha : \alpha < \omega_1\}$. In the construction of a Sierpinski set by transfinite induction at each stage α we can pick a point s_α outside of a given measure zero set $Z_\alpha \subset 2^\omega$. 2^ω has a natural structure of a topological group, and the sum of any two measure 1 sets is the whole group. Choose $s^0_\alpha, s^1_\alpha \in 2^\omega \setminus Z_\alpha$ such that $s^0_\alpha + s^1_\alpha = x_\alpha$ and $s^i_\alpha + \{s^{1-i}_\beta : \beta < \alpha\} \cap Q = \emptyset$. Set $S_i = \{s^i_\alpha : \alpha < \omega_1\}$. $\square$

Problem

- *Is it consistent that the product of two metrizable Menger spaces is Menger?*
- *Is it consistent that the product of two metrizable Hurewicz spaces is Hurewicz?*
- *Is it consistent that the product of two metrizable Hurewicz spaces is Menger?*

Theorem (Essentially A. Dow)

Let (X, τ) be a Lindelöf space. Then X is Menger in $V^{Fn(\mu,2)}$.

Proof. Two steps. 1. X remains Lindelöf. 2. X becomes Menger.

Proof of 1. Let $\dot{\mathcal{U}}$ be a $Fn(\mu, 2)$-name for an open cover of X by ground model open sets and $M \prec H(\theta)$ be such that $\dot{\mathcal{U}}, X, \mu, ... \in M$. Given any $x \in X$, consider
$D_x = \{p \in Fn(\mu, 2) \cap M : \exists U \in \tau \cap M \ (x \in U \wedge p \Vdash U \in \dot{\mathcal{U}}).\}$
D_x is dense in $Fn(\mu, 2) \cap M$: Fix $p \in Fn(\mu, 2) \cap M$ and for every $y \in X$ find $p_y \leq p$ and $y \in U_y \in \tau$ such that $p_y \Vdash U_y \in \dot{\mathcal{U}}$.
$\{U_y : y \in X\}$ is an open cover of X is V, so it contains a countable subcover $\{U_{y_n} : n \in \omega\}$, as witnessed by $\{p_n : n \in \omega\} \subset Fn(\mu, 2)$. By elementarity, we can assume $\{U_{y_n} : n \in \omega\}, \{p_n : n \in \omega\} \in M$, and hence $\{U_{y_n} : n \in \omega\} \cup \{p_n : n \in \omega\} \subset M$. Pick n such that $x \in U_{y_n}$ and note that $p_n \in D_x$.

Let G be $Fn(\mu, 2)$-generic. Then $H := G \cap M$ is $Fn(\mu, 2) \cap M$ generic.
$\dot{U}^G \cap M$ covers X: given $x \in X$, find $p \in D_x \cap H$ and $U \in \tau \cap M$ witnessing this, and note that $p \in G$ and $p \Vdash U \in \dot{\mathcal{U}}$, and hence $x \in U \in \dot{\mathcal{U}}^G$. □

Menger game

Game associated to Menger's property: In the n th move, I chooses an open cover $\mathcal{U}_n$ of X , and II responds by choosing $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$. Player II wins if $\{\cup\mathcal{V}_n : n \in \omega\}$ covers X. Otherwise, player I wins. A sequences $\langle \mathcal{U}_n, \mathcal{V}_n : n \leq \gamma \rangle$ is called a *play* in the Menger game, where $\gamma \leq \omega$.

Theorem (Hurewicz 192?)

X is Menger if and only if I has no winning strategy in the Menger game on X.

Proof. Sp-se X is Menger. Given a strategy F of I, we'll construct a play won by II, in which I uses F. Wlog, F instructs I to play with countable increasing covers. Set $F(\emptyset) = \mathcal{U}_\emptyset = \{U_{\langle n \rangle} : n \in \omega\}$ with $U_{\langle n \rangle} \subset U_{\langle n+1 \rangle}$ for all n. Sp-se II responds with $U_{\langle n \rangle}$. Then we set $F\langle U_{\langle n \rangle} \rangle = \{U_{\langle n,k \rangle} : k \in \omega\}$ and assume wlog $U_{\langle n,k \rangle} \subset U_{\langle n,k+1 \rangle}$ for all k. In general, given $\sigma = \langle n_i : i \leq m \rangle \in \omega^{m+1}$, it gives rise to a play

$$\Big\langle \mathcal{U}_\emptyset, U_{\langle n_0 \rangle}; F\langle U_{\langle n_0 \rangle} \rangle = \mathcal{U}_{\langle n_0 \rangle}, U_{\langle n_0, n_1 \rangle}; \ \ldots, \\ F\Big\langle U_{\langle n_0 \rangle}, \ldots, U_{\langle n_0, \ldots, n_{m-1} \rangle} \Big\rangle = \mathcal{U}_{\langle n_0, \ldots n_{m-1} \rangle}, U_{\langle n_0, \ldots n_{m-1}, n_m \rangle} = U_\sigma \Big\rangle$$

in which I uses F, and the next response of I is $\mathcal{U}_\sigma = \{U_{\sigma\frown k} : k \in \omega\}$ with $U_{\sigma\frown k} \subset U_{\sigma\frown\langle k+1 \rangle}$. Wlog, $U_\sigma = U_{\sigma\frown 0}$.

Let $\mathcal{O}_n = \left\{O^n_k = \bigcap_{\sigma\in\omega^{\uparrow n+1},\sigma(n)=k} U_\sigma \;:\; k\in\omega\right\}$. $\mathcal{O}_n$ covers X: If not, pick x and $\langle\sigma_k : k\in\omega\rangle\subset\omega^{\uparrow(n+1)}$ such that $\sigma_k(n)=k$ and $x\notin U_{\sigma_k}$. Let $m=\min\left\{i : \{\sigma_k(i) : k\in\omega\}\text{ is unbounded}\right\}$. Let $K\in[\omega]^\omega$ be s.t. $\tau=\sigma_k\restriction m$ is the same for all $k\in K$ and $\sigma_{k_0}(m)<\sigma_{k_1}(m)$ for all $k_0<k_1$ in K. Then $U_{\sigma_k\restriction(m+1)}=U_{\tau\,\hat{}\,\sigma_k(m)}$ for all $k\in K$, and so $\{U_{\sigma_k\restriction(m+1)} : k\in K\}$ covers X, being cofinal in $\mathcal{U}_\tau$. But $U_{\sigma_k}\supset U_{\sigma_k\restriction(m+1)}$, and hence $\{U_{\sigma_k} : k\in K\}$ covers X, a contradiction

Let $f\in\omega^{\uparrow\omega}$ be such that $\bigcup_{n\in\omega} O^n_{f(n)}=X$. Look at the play $\langle\mathcal{U}_\emptyset, U_{\langle f(0)\rangle}; \ldots, \mathcal{U}_{f\restriction n}, U_{f\restriction n\,\hat{}\,f(n)} = U_{f\restriction(n+1)}; \ldots\rangle$. Since $U_{f\restriction(n+1)}\supset O^n_{f(n)}$, this play is lost by I. $\square$

A space (X,τ) is called a *D-space*, if for every $f : X\to\tau$ such that $x\in f(x)$ for all x, there exists a closed discrete $D\subset X$ such that $X=\bigcup_{x\in D} f(x)$.

Problem

Is every regular Lindelöf space a D-space?

Menger spaces are D-spaces (Aurichi 2010).

Let f be a neighbourhood assignment. Consider the following strategy of I in the Menger game on X. $\mathcal{U}_\emptyset = \{f(x) : x \in X\}$. Suppose that II replies with $\{f(x) : x \in F_0\}$ for some $F_0 \in [X]^{<\omega}$. Letting $U_0 = \bigcup\{f(x) : x \in F_0\}$, I suggests $\{U_0\} \cup \{f(x) : x \in X \setminus U_0\}$. Suppose that II replies with $\{U_0\} \cup \{f(x) : x \in F_1\}$ for some $F_1 \in [X \setminus U_0]^{<\omega}$. Letting $U_1 = \bigcup\{f(x) : x \in F_1\}$, I suggests $\{U_0, U_1\} \cup \{f(x) : x \in X \setminus (U_0 \cup U_1)\}$. Suppose that II replies with $\{U_0, U_1\} \cup \{f(x) : x \in F_2\}$ for some $F_2 \in [X \setminus (U_0 \cup U_1)]^{<\omega}$. Letting $U_2 = \bigcup\{f(x) : x \in F_2\}$, I suggests $\{U_0, U_1, U_2\} \cup \{f(x) : x \in X \setminus (U_0 \cup U_1 \cup U_2)\}$, and so on.

There is a play lost by I, which yields a sequence $\langle U_n = \bigcup_{x \in F_n} f(x) : n \in \omega\rangle$ covering X s.t. $F_{n+1} \subset X \setminus \bigcup_{i \leq n} U_n$. $\bigcup_{n \in \omega} F_n$ is a closed discrete kernel of f. $\square$

The last slide

Thank you for your attention.