Ultrafilters on semifilters

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A semifilter is, for our purposes, an upward-closed subset of the partial order $\mathcal{P}(\omega)/\mathrm{fin}$. In other words, a semifilter is like a (free) filter except that it is not required to be closed under finite intersections. Being subsets of $\mathcal{P}(\omega)/\mathrm{fin}$, semifilters are partial orders. The (ultra)filters on a given semifilter $\mathfrak{S}$ correspond to closed subsets of ω^* (via Stone duality), and these subsets can have interesting dynamical/algebraic properties for certain choices of $\mathfrak{S}$.

The cardinal numbers $\mathfrak{p}$ and $\mathfrak{t}$ describe combinatorial aspects of the partial order $\mathcal{P}(\omega)/\mathrm{fin}$. Analogous constants can be defined for any partial order $\mathbb{P}$: its pseudo-intersection number $\mathfrak{p}_{\mathbb{P}}$ and its tower number $\mathfrak{t}_{\mathbb{P}}$. We will show that if $\mathfrak{S}$ is a semifilter that is G_δ in 2^ω, then $\mathfrak{p}_{\mathfrak{S}} = \mathfrak{t}_{\mathfrak{S}} = \mathfrak{p}$.[1] If $\mathfrak{p} = \mathfrak{c}$ then this allows us to build ultrafilters on G_δ semifilters that are also P-filters, and we will discuss some consequences of this for the dynamical/algebraic structure of ω^*.

[1]Since $\mathcal{P}(\omega)/\mathrm{fin}$ is G_δ in 2^ω, this result implies the Malliaris-Shelah equality $\mathfrak{p} = \mathfrak{t}$. We are not claiming to have found a new proof of this equality. Instead, will show $\mathfrak{p} \leq \mathfrak{p}_{\mathfrak{S}} \leq \mathfrak{t}_{\mathfrak{S}} \leq \mathfrak{t}$ and then use Malliaris and Shelah's equality to complete our proof.