Ordered sets of Baire class 1 functions

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joint work with Márton Elekes
General question

Pointwise ordering

Let \( X \) be an uncountable Polish space and \( \mathcal{F} \) a set of functions \( f : X \to \mathbb{R} \).

Definition.
For \( f, g \in \mathcal{F} \) we say that \( f < g \) if for every \( x \in X \) we have \( f(x) \leq g(x) \) and there exists an \( x \) such that \( f(x) < g(x) \).

The general question

Let \((L, \prec_L)\) be an ordering. Does there exist an (order preserving) embedding \((L, \prec_L), \to (\mathcal{F}, \prec)\)?

Definition.
Suppose that \((P, \prec_P)\) and \((Q, \prec_Q)\) are posets. We say that \( P \) is embeddable into \( Q \), in symbols \((P, \prec_P), \to (Q, \prec_Q)\), if there exists a map \( \Phi : P \to Q \) such that for every \( p, q \in P \) if \( p \prec_P q \) then \( \Phi(p) \prec_Q \Phi(q) \).
General question

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Let $(\mathbb{L}, <_{\mathbb{L}})$ be an ordering. Does there exist an (order preserving) embedding $(\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (\mathcal{F}, <)$?
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**Definition.** Suppose that $(P, <_P)$ and $(Q, <_Q)$ are posets. We say that $P$ is **embeddable into** $Q$, in symbols $(P, <_P) \hookrightarrow (Q, <_Q)$ if there exists a map $\Phi : P \to Q$ such that for every $p, q \in P$ if $p <_P q$ then $\Phi(p) <_Q \Phi(q)$. 
Known results: Continuous case

**Proposition.** (Folklore) For a linearly ordered set \((\mathbb{L}, <_\mathbb{L})\)

\[(\mathbb{L}, <_\mathbb{L}) \hookrightarrow (C(X, \mathbb{R}), <) \text{ iff } (\mathbb{L}, <_\mathbb{L}) \hookrightarrow ([0, 1], <).\]

In fact,

\[(C(X, \mathbb{R}), <) \Leftrightarrow ([0, 1], <).\]
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**Proposition.** (Folklore) For a linearly ordered set \((L, \leq_L)\)

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In fact,

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**Proof.**

\([0, 1], \leq) \hookrightarrow (C(X, \mathbb{R}), \leq):\text{ obvious.}\]
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\((C(X, \mathbb{R}), <) \hookrightarrow ([0, 1], <): \text{ the set of closed sets of a Polish space } Y \text{ (denoted by } \Pi^0_1(Y)\text{) forms a poset with the strict inclusion.}\)
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The map \(f \mapsto \text{subgraph}(f) = \{(x, y): y \leq f(x)\}\) is an embedding

\((C(X, \mathbb{R}), <) \hookrightarrow (\Pi^0_1(X \times \mathbb{R}), \subset).\)
Known results: Continuous case

Enough:

\[(\Pi_1^0(X \times \mathbb{R}), \subseteq) \hookrightarrow ([0, 1], <)\]

Let \(\{U_n : n \in \omega\}\) be a basis of \(X \times \mathbb{R}\).
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Let \( \{U_n : n \in \omega\} \) be a basis of \( X \times \mathbb{R} \).

Map \( F \in \Pi_1^0(X \times \mathbb{R}) \) to \( \sum_{U_n \cap F \neq \emptyset} 3^{-n-1} \).

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Observe that we did not use the continuity, just that the sets subgraph\((f)\) are closed.

**Definition.** A function \(f\) is called *upper semicontinuous (USC)* if subgraph\((f)\) is closed.
Known results: Higher Baire classes

Borel functions

**Theorem.** (Komjáth, 1990) The existence of $\omega_2 \rightarrow (\mathcal{B}(X), <)$ is already independent of ZFC.
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**Definition.** Let $\xi < \omega_1$ and $\mathcal{B}_0(X) = C(X, \mathbb{R})$. A function is called a *Baire class* $\xi$ function (i. e. it is the element of $\mathcal{B}_\xi(X)$) if it is the pointwise limit of functions that are all in Baire classes of indices less than $\xi$. 
Known results: Higher Baire classes

Baire class 2 functions

**Theorem.** *(Komjáth, 1990)* The existence of \( \omega_2 \rightarrow (\mathcal{B}_2(X), <) \) is already independent of ZFC.

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Known results: Baire class 1

Kuratowski’s theorem

**Theorem.** (Kuratowski, 60s) $\omega_1$ and $\omega_1^*$ are not embeddable in $(\mathcal{B}_1(X), <)$.
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**Theorem.** (Komjáth, 1990) Consistently no: If $(\mathcal{S}, <)$ is a Suslin line, then $(\mathcal{S}, <) \not\subset (\mathcal{B}_1(X), <)$. 

Known results: Baire class 1

A non-characterisation result

**Theorem.** (Elekes, Steprāns, 2006) There exists a linear ordering $(\mathbb{L}, <_{\mathbb{L}})$ such that neither $\omega_1$ nor $\omega^*_1$ is embeddable into $\mathbb{L}$, but $(\mathbb{L}, <_{\mathbb{L}}) \not\rightarrow (B_1(X), <)$. 
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The positive direction

**Theorem.** (Elekes, Steprāns, 2006) (MA) If \(|\mathbb{L}| < c\) and neither \(\omega_1\) nor \(\omega_1^*\) is embeddable into \((\mathbb{L}, <_{\mathbb{L}})\) then \((\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (\mathcal{B}_1(X), <)\).
Main question

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**Main Theorem.** (Elekes, V.) There exists a universal linear ordering embeddable into the poset of Baire class 1 functions, i.e., a linearly ordered set \((U, <_U)\) such that for every linearly ordered set \((\mathbb{L}, <_{\mathbb{L}})\) we have

\[(\mathbb{L}, <_{\mathbb{L}}) \leftrightarrow (\mathcal{B}_1(X), <) \iff (\mathbb{L}, <_{\mathbb{L}}) \leftrightarrow (U, <_U).\]

The universal ordering: \( ([0, 1]^{< \omega_1}, <_{altlex}) \)

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in \([0, 1]\) with last element 0 by \([0, 1]^{< \omega_1}\).
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Let \(\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0, 1]^{<\omega_1}\) be distinct and let \(\delta\) be minimal such that \(x_\delta \neq x'_\delta\). We say that \(\bar{x} <_{\text{altlex}} \bar{x}' \iff \)
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\[ x_\delta < x'_\delta \text{ if } \delta \text{ is even or } \]

\[ x_\delta > x'_\delta \text{ if } \delta \text{ is odd.} \]

Main Theorem. (Elekes, V.)

\(\langle L, < \rangle \rightarrow \langle B_1(X), < \rangle\) iff \(\langle L, < \rangle \rightarrow \langle [0, 1]^{< \omega_1}, < \text{altlex} \rangle\).
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Main Theorem. (Elekes, V.) \([L, <_L] \rightarrow (B_1(X), <)\) iff \([L, <_L] \rightarrow ([0, 1]^{<\omega_1}, <_{altlex})\). In fact, \((B_1(X), <) \rightarrow \rightarrow ([0, 1]^{<\omega_1}, <_{altlex})\).
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In fact,

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About the proof

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$(B_1(X), <) \leftrightarrow ([0, 1]^{<\omega_1}, <_{\text{altlex}})$

**Theorem.** (Hausdorff, Kuratowski) For every $A \in \Delta^0_2$ there exists a strictly decreasing transfinite sequence of closed sets $(F_\beta)_{\beta \leq \xi}$ for some $\xi < \omega_1$ such that

$$A = \bigcup_{\gamma < \xi, \gamma \text{ is even}} (F_\gamma \setminus F_{\gamma + 1}).$$
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$$( [0, 1]^{<\omega_1}, <_{altlex} ) \leftrightarrow ( B_1(X), < )$$

$X, X'$ are $\sigma$-compact then $(B_1(X), <) \equiv (B_1(X'), <)$.

Enough: $([0, 1]^{<\omega_1}, <_{altlex}) \leftrightarrow (\Delta^0_2(K([0, 1]^2)), \subset)$. 
Applications: New proofs of old results

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- Elekes-Steprāns: a special Aronszajn-line is embeddable.
- Komjáth: a forcing-free proof of the non-embeddability of Suslin lines.
Applications: New results

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Lexicographical countable products of embeddable linearly ordered sets are also embeddable. Completions of an embeddable linearly ordered set are not necessarily embeddable.
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Open problems

**Question.** What can we say about linear orderings embeddable into the poset of Baire class \( \alpha \) functions if \( \alpha \geq 2 \) in terms of universal orderings? What if we consider the poset \( (\Sigma^0_\alpha(X), \subset) \) for some \( \alpha \geq 2 \)?
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**Question.** Does there exist an embedding $(\mathcal{B}_1(X), <) \leftrightarrow (\Delta^0_2(X), \subseteq)$ such that $(\mathcal{B}_1(X), <)$ is (as a poset) isomorphic to its image?
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**Question.** Does there exist a universal linearly ordered set if $X$ is only separable metrisable?
Thank you for your attention!