Remainders of topological groups and ultrafilters

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Note that these theorems are dichotomies.

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σ-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger $\Rightarrow$ Lindelöf.

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A space X is Hurewicz iff for any Čech-complete Z containing X as a dense subspace, there exists a σ-compact F such that $X \subset F \subset Z$. $\square$.

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If $\mathfrak{d} = \mathfrak{c}$, then there exists an ultrafilter $\mathcal{U}$ such that the Mathias forcing $\mathbb{M}_{\mathcal{U}}$ associated to it does not add dominating reals.

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For an ultrafilter $\mathcal{U}$ on ω, the Mathias forcing $\mathbb{M}_{\mathcal{U}}$ associated to it does not add dominating reals iff $\mathcal{U}$ is Scheepers

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Every Scheepers (equiv. Menger ultrafilter) is a P-point.

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The existence of a topological group G such that $\beta G \setminus G$ is Scheepers and not σ-compact is independent from ZFC.

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For a semifilter $\mathcal{F}$ we denote by $\mathbb{P}_\mathcal{F}$ the poset consisting of all partial maps p from $\omega \times \omega$ to 2 such that for every $n \in \omega$ the domain of $p_n : k \mapsto p(n,k)$ is an element of $\sim \mathcal{F} := \{\omega \setminus F : F \in \mathcal{F}\}$.

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A condition q is stronger than p (in this case we write $q \leq p$) if $p \subset q$.

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The last slide

Thank you for your attention.