Remainders of topological groups and ultrafilters

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**Question**: What could be said about remainders? I.e., do they have some special interesting properties?
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Let $G$ be a topological group and $bG$ a compactification of $G$. If $Y = bG \setminus G$, then $Y$ is either Lindelöf or pseudocompact.
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Note that these theorems are dichotomies.
A Lindelöf topological space $X$ is called
- *Hurewicz,*
- *Scheepers,*
- *Menger,*

If $X$ is zero-dimensional, then it is enough to consider functions into $\omega^\omega$. 

$\sigma$-compact $\Rightarrow$ Hurewicz $\Rightarrow$ Scheepers $\Rightarrow$ Menger $\Rightarrow$ Lindelöf.
A Lindelöf topological space $X$ is called
- **Hurewicz**, if for any continuous $f : X \to \mathbb{R}^\omega$, the range $f[X]$ is bounded;
- **Scheepers**, if for any continuous $f : X \to \mathbb{R}^\omega$, the collection $\{\max A : A \in f[X]\} < \omega$ is not dominating;
- **Menger**, if for any continuous $f : X \to \mathbb{R}^\omega$, the range $f[X]$ is not dominating.

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Between $\sigma$-compact and Lindelöf: combinatorial properties

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A space $X$ is Hurewicz iff for any Čech-complete $Z$ containing $X$ as a dense subspace, there exists a $\sigma$-compact $F$ such that $X \subset F \subset Z$. □.
The Scheepers property

Do the properties of Scheepers and Menger also imply being $\sigma$-compact for remainders of topological groups?

Observation

There exists a topological group $G$ such that $\beta G \setminus G$ is Scheepers and not $\sigma$-compact if there exists a Scheepers ultrafilter $U$ on $\omega$.

Proof. Let $U^*$ be the dual ideal. $(U^*, \Delta)$ is a topological group and $U^* \cup U = \mathcal{P}(\omega)$.

Theorem (Canjar 1988)

If $d = c$, then there exists an ultrafilter $U$ such that the Mathias forcing $M_U$ associated to it does not add dominating reals.

Theorem (Chodounsky-Repovs-Z. 2014)

For an ultrafilter $U$ on $\omega$, the Mathias forcing $M_U$ associated to it does not add dominating reals if $U$ is Scheepers if $U$ is Menger.
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Theorem (Canjar 1988)

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Theorem (Chodounsky-Repovs-Z. 2014)

*For an ultrafilter \(\mathcal{U}\) on \(\omega\), the Mathias forcing \(\mathbb{M}_\mathcal{U}\) associated to it does not add dominating reals iff \(\mathcal{U}\) is Scheepers*
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There exists a topological group $G$ such that $\beta G \setminus G$ is Scheepers and not $\sigma$-compact if there exists a Scheepers ultrafilter $\mathcal{U}$ on $\omega$.

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*For an ultrafilter \( \mathcal{U} \) on \( \omega \), the Mathias forcing \( \mathbb{M}_\mathcal{U} \) associated to it does not add dominating reals iff \( \mathcal{U} \) is Scheepers iff \( \mathcal{U} \) is Menger.*
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Every Scheepers (equiv. Menger ultrafilter) is a $P$-point.
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Every Scheepers (equiv. Menger ultrafilter) is a $P$-point.

Corollary
The existence of a topological group $G$ such that $\beta G \setminus G$ is Scheepers and not $\sigma$-compact is independent from ZFC.
The Menger property

Theorem

It is consistent that for any topological group $G$ and compactification $bG$, if $(bG \setminus G)^2$ is Menger, then it is $\sigma$-compact. 

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The proof uses the following forcing:
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The proof uses the following forcing:
For a semifilter $\mathcal{F}$ we denote by $\mathbb{P}_\mathcal{F}$ the poset consisting of all partial maps $p$ from $\omega \times \omega$ to 2 such that for every $n \in \omega$ the domain of $p_n : k \mapsto p(n, k)$ is an element of $\sim \mathcal{F} := \{\omega \setminus F : F \in \mathcal{F}\}$. 
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If, moreover, we assume that and $\text{dom}(p_n) \subset \text{dom}(p_{n+1})$ for all $n$, the corresponding poset will be denoted by $\mathbb{P}^*_\mathcal{F}$.
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If, moreover, we assume that and $\text{dom}(p_n) \subset \text{dom}(p_{n+1})$ for all $n$, the corresponding poset will be denoted by $\mathbb{P}_\mathcal{F}^*$.

A condition $q$ is stronger than $p$ (in this case we write $q \leq p$) if $p \subset q$. 
The poset, continued

For filters $F$ for the poset $P$, $P^F$ is obviously dense in $P$, and the latter is well-known to be proper and $\omega$-bounding if $F$ is a non-meager $P$-filter.

$F^+ = \{ X \subset \omega : \forall F \in F (X \cap F \neq \emptyset) \}$

Lemma. If $F^+$ is a Menger semilter, then both $P^F$ and $P^*F$ are proper and $\omega_\omega$-bounding.

Example. $F = [\omega]$ $\omega$. Then $F^+$ is the Frechet filter $\{ \omega \setminus A : A \in [\omega]^{<\omega} \}$, hence Menger (even countable). Then $P^F$ is proper and $\omega_\omega$-bounding.

Note that it is the full support product of countably many Silver forcings.
For filters $\mathcal{F}$ the poset $\mathbb{P}^*_\mathcal{F}$ is obviously dense in $\mathbb{P}_\mathcal{F}$, and the latter is well-known to be proper and $\omega^\omega$-bounding if $\mathcal{F}$ is a non-meager $P$-filter.
For filters $\mathcal{F}$ the poset $\mathbb{P}^*_F$ is obviously dense in $\mathbb{P}_F$, and the latter is well-known to be proper and $\omega^\omega$-bounding if $\mathcal{F}$ is a non-meager $P$-filter.

$\mathcal{F}^+ = \{X \subset \omega : \forall F \in \mathcal{F} (X \cap F \neq \emptyset)\}$

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*If $\mathcal{F}^+$ is a Menger semifilter, then both $\mathbb{P}_\mathcal{F}$ and $\mathbb{P}_\mathcal{F}^*$ are proper and $\omega^\omega$-bounding.*
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**Example.** $\mathcal{F} = [\omega]^{<\omega}$. Then $\mathcal{F}^+$ is the Frechét filter

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**Example.** $\mathcal{F} = [\omega]^{<\omega}$. Then $\mathcal{F}^+$ is the Frechét filter $\{ \omega \setminus A : A \in [\omega]^{<\omega} \}$, hence Menger (even countable). Then $\mathbb{P}_\mathcal{F}$ is proper and $\omega^\omega$-bounding. Note that it is the full support product of countably many Silver forcings.
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For filters \( \mathcal{F} \) the poset \( \mathbb{P}^*_\mathcal{F} \) is obviously dense in \( \mathbb{P}_\mathcal{F} \), and the latter is well-known to be proper and \( \omega^\omega \)-bounding if \( \mathcal{F} \) is a non-meager \( P \)-filter.

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If $\mathcal{F}$ is the Frechét filter, hence $\sim \mathcal{F} = [\omega]^{< \omega}$, i.e., $\mathbb{P}_\mathcal{F}$ is the countably supported product of the Cohen forcing, and therefore $\mathbb{P}_\mathcal{F}$ collapses $(2^\omega)^V$ to $\omega$. 

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Questions

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Is there a ZFC example of a topological group with a Menger non-σ-compact remainder?
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Suppose that \(\beta(C_p(X)) \setminus C_p(X)\) is Menger. Is it then \(\sigma\)-compact? Equivalently, is \(X\) countable discrete?
Thank you for your attention.