Resurrection axioms and generic absoluteness

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Hilbert’s Program (finding a complete and consistent theory for mathematics) had an abrupt stop after Gödel’s Incompleteness Theorem in 1931.

The question whether it would be possible to have **empiric completeness** was left open, even if early results showed that ZFC does not have such a behavior.

Many of this results are obtained using forcing, thus in order to achieve empiric completeness we need to “rule it out”. How?

**Definition**

A theory $T$ has **generic absoluteness** for a family $\Theta$ of first-order formulas and a definable class $\Gamma$ of CBAs iff in all models of $T$ the truth values of formulas in $\Theta$ cannot be changed in forcing extensions obtained by CBAs in $\Gamma$ which preserves $T$. 
We shall base on boolean valued models approach to forcing, and consider the following classes $\Gamma$ of CBAs defined by properties interesting for forcing:

- $\Omega$, the class of all CBAs,
- $\kappa$-distributive, $\kappa$-cc
- axiom-$A$, proper, semiproper (SP),
- stationary set preserving (SSP).

We shall equip a class $\Gamma$ with two partial orders:

- $\mathbb{B} \leq_{\Gamma} \mathbb{C}$ iff there exists a complete homomorphism $i : \mathbb{C} \to \mathbb{B}$ such that the quotient algebra $\mathbb{B}/i[\dot{G}_C]$ is in $\Gamma$ with boolean value $1_C$,
- $\mathbb{B} \leq^*_{\Gamma} \mathbb{C}$ iff there exists a complete injective homomorphism as above.

We denote by $\mathcal{U}_{\kappa}^\Gamma$ (category forcing) the set $\Gamma \cap H_\kappa$ ordered by $\leq_{\Gamma}$.

We say that $\Gamma$ is iterable iff it is closed under two-step iterations, lottery sums and the order $\leq^*_{\Gamma}$ is closed for set-sized descending sequences of elements of $\Gamma$. Most of the interesting classes are iterable, with the notable exception of SSP.
Definition

A theory $T$ has *generic absoluteness* for a family $\Theta$ of first-order formulas and a definable class $\Gamma$ of CBAs iff in all models of $T$ the truth values of formulas in $\Theta$ cannot be changed in forcing extensions obtained by CBAs in $\Gamma$ which preserves $T$.

Fundamental generic absoluteness results are known in the literature for ZFC with large cardinals, e.g:

- ZFC: generic absoluteness for $\Gamma = \Omega$ and $\Theta = \Sigma_2^1(\mathbb{R})$ (*Shönfield*)
- ZFC + $\exists$class many Woodin cardinals limit of Woodin cardinals: generic absoluteness for $\Gamma = \Omega$ and $\Theta$ the formulas with real parameters relativized to $L(\mathbb{R})$ (*Woodin*)

More generic absoluteness results can be obtained in extensions of ZFC with forcing axioms.
Intuitively, forcing axioms postulate that $V$ is closed under taking suitable forcing extensions over small set models of ZFC.

**Definition**

$\text{FA}(\Gamma)$ states that for all $B \in \Gamma$ and collection $D$ of $\aleph_1$-many dense subsets of $B$, there exists a filter $F$ meeting all of them.

Note that the same sentence for $\aleph_0$-many dense subsets is Baire’s Category Theorem.

Other variations we will consider are $\text{BFA}(\Gamma)$ (weakening) and $\text{FA}^{++}(\Gamma)$ (strengthening). Recall that MM, PFA are shorthands for FA(SSP), FA(proper).
Many of the commonly used forcing axiom can be restated as density properties (under suitable large cardinal hypothesis):

- **BFA(Γ)** holds iff the class \( \{ B \in \Gamma : H_{\aleph_2} \prec_1 V^B \} \) is dense in \((\Gamma, \leq_{all})\),
- **FA(SSP)** holds iff \( \{ B \in SSP : B \text{ is presaturated} \} \) is dense in \((SSP, \leq_{all})\) (class many Woodin cardinals),
- **FA^{++}(SSP)** holds iff the same class is dense in \((SSP, \leq_{SSP})\) (class many Woodin cardinals),
- **MM^{+++}** holds iff \( \{ B \in SSP : B \text{ is strongly presaturated} \} \) is dense in \((SSP, \leq_{SSP})\).
Examples of generic absoluteness results known in literature for extensions of ZFC with forcing axioms are:

- BFA($\Gamma$) is equivalent to ZFC having generic absoluteness for $\Theta$ the $\Sigma_1$ formulas with parameters relativized to $H_{\aleph_2}$ and CBAs in $\Gamma$ (Bagaria),
- $\text{ZFC} + \text{MM}^{+++} + \exists$ class many superhuge cardinals has generic absoluteness for $\Gamma = \text{SSP}$ and $\Theta$ the formulas relativized to $L([\text{ON}]^{\aleph_1})$ (Viale).

We show that strong generic absoluteness results can be obtained from resurrection axioms (of lower consistency strength).
We can develop the definition of the resurrection axiom starting from a model-theoretic point of view.

**Theorem**

Let $M \subset N$ be models of a language $\mathcal{L}$. Then TFAE:

- $M$ is existentially closed in $N$ ($M \prec_1 N$),
- $M$ has resurrection, i.e. it exists a larger $M' \supseteq N$ such that $M \prec M'$

If we restrict the above properties to models of set theory of the form $H^M_c$ where $c = \aleph_2$ and consider only model extensions obtained by forcing in a fixed class $\Gamma$, we obtain respectively:

- $M$ satisfies BFA($\Gamma$),
- $M$ satisfies RA($\Gamma$), the resurrection axiom
Resurrection axioms have been introduced recently by Hamkins and Johnstone, and are interesting since they can prove some consequences of FA, while having much lower consistency strength (for $\Gamma \neq \text{SSP}$).

In particular, we have that:

- $\text{RA}(\Gamma)$ for all mentioned $\Gamma$ implies that $c \leq \aleph_2$,
- $\text{RA}(\Gamma) + \neg \text{CH}$ implies $\text{BFA}(\Gamma)$,
- $\text{FA}(\Gamma)$ is consistent relative to a supercompact cardinal ($\text{Foreman, Magidor, Shelah}$),
- $\text{RA}(\Gamma)$ for iterable $\Gamma$ is consistent relative to a Mahlo cardinal ($\text{Hamkins, Johnstone}$),
- $\text{RA}(\text{SSP})$ is consistent relative to a inaccessible limit of Woodin cardinals above a supercompact cardinal ($\text{Asperó}$).
The resurrection axiom is conveniently stated as a density property:

**Definition**

RA(Γ) holds iff the class \( \{ B \in \Gamma : H_c \prec H_c^{V^B} \} \) is dense in \( (\Gamma, \preceq_{\Gamma}) \).

From RA(Γ) we can already prove a weak form of generic absoluteness:

**Theorem (Viale)**

ZFC + RA(Γ) has generic absoluteness for \( \Theta \) the \( \Sigma_2 \) formulas with parameters relativized to \( H_c \) and forcing in \( \Gamma \).

To achieve a stronger generic absoluteness result we need a stronger definition.
Definition (iterated resurrection axiom)

$\text{RA}_\omega(\Gamma)$ postulates that it is possible to resurrect the theory of $H_c$ any fixed finite number of times.

Precisely, $\text{RA}_\alpha(\Gamma)$ is the assertion:

$$\forall \beta < \alpha \text{ and } \forall N \supseteq M \text{ obtained by forcing in } \Gamma,$$
$$\exists M' \supseteq N \text{ a further extension by a forcing in } \Gamma,$$

such that $H^M_c < H^{M'}_c$ and $M'$ satisfies $\text{RA}_\beta(\Gamma)$.

\[
\begin{array}{ccc}
\text{RA}_\alpha(\Gamma) & \text{RA}_\beta(\Gamma) & \text{RA}_\gamma(\Gamma) \\
M & \overset{\Sigma_\omega(H_c)\text{-elem.}}{\rightarrow} & M' & \overset{\Sigma_\omega(H_c)\text{-elem.}}{\rightarrow} & M'' \\
\downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma & \downarrow \Gamma \\
N & \rightarrow & N' & \rightarrow & \ldots
\end{array}
\]

$\alpha > \beta > \gamma > \ldots$
Also the iterated resurrection axiom is conveniently stated as a density property:

**Definition**

\( \text{RA}_\alpha(\Gamma) \) holds iff for all \( \beta < \alpha \) the class

\[
\left\{ \mathcal{B} \in \Gamma : H_c \prec H^V_{\mathcal{B}} \land V^\mathcal{B} \models \text{RA}_\beta(\Gamma) \right\}
\]

is dense in \((\Gamma, \leq_\Gamma)\).
From this strengthened axiom we can obtain:

**Theorem (A., Viale)**

\[ \text{ZFC} + \text{RA}_\omega(\Gamma) \text{ has generic absoluteness for } \Theta \text{ the formulas relativized to } H_\text{c} \text{ and forcing in } \Gamma. \]

This directly improves the generic absoluteness result about \( \text{ZFC} + \text{RA}(\Gamma) \), whereas with respect to Viale’s absoluteness about \( \text{ZFC} + \text{MM}^{+++} + \text{LC} \):

- \( \Theta \) is smaller since \( H_\text{c} \subset L([\text{ON}]^{\aleph_1}) \),
- it is more general since it holds for any \( \Gamma \) (not only SSP),
- it has lower consistency strength
Lemma

$\text{ZFC} + \text{RA}_n(\Gamma)$ has generic absoluteness for $\Theta$ the $\Sigma_{n+1}$ formulas relativized to $H_c$ and forcing in $\Gamma$.

Proof.

By induction on $n$, consider a $\Sigma_{n+1}$ formula $\phi = \exists x \psi(x)$ and draw the following:

- $M \models \psi^{H_c}(a) \Rightarrow N \models \psi^{H_c}(a)$ so $M \models \exists x \psi^{H_c}(x) \Rightarrow N \models \exists x \psi^{H_c}(x)$,
- $N \models \exists x \psi^{H_c}(x) \Rightarrow M' \models \exists x \psi^{H_c}(x)$ (same argument) $\Rightarrow M \models \exists x \psi^{H_c}(x)$ (elementarity).
**Theorem (A., Viale)**

The following holds:
- $\text{RA}_{\text{ON}}(\Gamma)$ for iterable $\Gamma$ is consistent relative to a Mahlo cardinal,
- $\text{RA}_{\text{ON}}(\text{SSP})$ is consistent relative to a stationary limit of supercompact cardinals,
- $\text{MM}^{+++} \Rightarrow \text{RA}_{\text{ON}}(\text{SSP})$.

**Sketchy proof.**

To prove consistency of $\text{RA}_\alpha(\Gamma)$ with $\Gamma$ iterable (as for $\text{FA}(\Gamma)$ and variations), we use lottery iteration forcing with respect to suitable fast-growing (Menas) function $f : \kappa \to \kappa$ for a large enough cardinal $\kappa$.

\[
\begin{align*}
\mathcal{B}_0 &= 2 \\
\mathcal{B}_{\alpha+1} &= \mathcal{B}_\alpha \ast \mathcal{C}_\alpha \text{ where } \mathcal{C}_\alpha = \prod \left( \Gamma \cap H^{V_{\mathcal{B}_\alpha}}_{f(\alpha)} \right) \\
\mathcal{B}_\alpha \text{ for } \alpha \text{ limit is a lower bound in } \Gamma \text{ for the chain } \langle \mathcal{B}_\beta : \beta < \alpha \rangle
\end{align*}
\]

For $\Gamma = \text{SSP}$ we use the category forcing $\bigcup_{\kappa}^{\text{SSP}}$ for a large enough cardinal $\kappa$. \qed
We developed a strengthening \( RA_{\alpha}(\Gamma) \) of \( RA(\Gamma) \). Other strengthened forms have been developed:

**Definition (Hamkins, Johnstone)**

The *boldface resurrection axiom* \( RA(\Gamma) \) holds iff for every \( A \subseteq c \) the following class:

\[
\{ B \in \Gamma : \exists A^* \subseteq c^B \langle H^c, \in, A \rangle \prec \langle H^B, \in, A^* \rangle \}
\]

is dense in \((\Gamma, \leq_{\Gamma})\).

Its consistency can be proved from a strongly uplifting cardinal (below subtle and \( 0^\# \), above weakly compact).

For all classes \( \Gamma \), \( RA(\Gamma) \) implies \( RA(\Gamma) \).

It does not imply \( RA_\omega(\Gamma) \).
We developed a strengthening $\text{RA}_\alpha(\Gamma)$ of $\text{RA}(\Gamma)$. Other strengthened forms have been developed:

**Definition (Tsaprounis)**

The *unbounded resurrection axiom* $\text{UR}(\Gamma)$ holds iff for every $\theta > c$ the class

$$\{ B \in \Gamma : \exists j : H_\theta \rightarrow H_{j(\theta)}^{V^B}, \text{crit}(j) = c \land j(c) > \theta \}$$

is dense in $(\Gamma, \leq_\Gamma)$ below $\Gamma \cap H_\theta$, and $c = \omega_2$.

Its consistency can be proved from an extendible cardinal (higher consistency strength than $\text{FA}(\Gamma)$).

For most classes $\Gamma$, $\text{UR}(\Gamma)$ implies $\text{FA}^{++}(\Gamma)$ and $\text{RA}(\Gamma)$. It does not imply $\text{RA}_\omega(\Gamma)$. 
It is also possible to combine the iteration idea of \(\text{RA}_\alpha(\Gamma)\) with the boldface version \(\text{RA}(\Gamma)\).

**Definition (A., Hamkins, Johnstone, Viale)**

The **boldface \(\alpha\)-resurrection axiom** \(\text{RA}_\alpha(\Gamma)\) holds iff for every \(A \subseteq c\) and \(\beta < \alpha\) the following class:

\[
\left\{ B \in \Gamma : \exists A^* \subseteq c^B \langle H_c, \in, A \rangle \prec \langle H^B_c, \in, A^* \rangle \land V^B \models \text{RA}_\beta(\Gamma) \right\}
\]

is dense in \((\Gamma, \leq_\Gamma)\).

Its consistency can be proved from an \(\alpha\)-strongly uplifting cardinal (like \(\text{RA}(\Gamma), \sim\) below \(0^\#\) and above weakly compact).

For all classes \(\Gamma\), \(\text{RA}_\alpha(\Gamma)\) implies \(\text{RA}_\alpha(\Gamma)\).
Definition

The unbounded $\alpha$-resurrection axiom $\text{UR}_\alpha(\Gamma)$ holds iff for every $\theta > c$ and $\beta < \alpha$ the class

$$
\left\{ B \in \Gamma : \exists j : H_\theta \rightarrow H_{j(\theta)}^B \text{ crit}(j) = c \land j(c) > \theta \land V^B \models \text{UR}_\beta(\Gamma) \right\}
$$

is dense in $(\Gamma, \leq_\Gamma)$ below $\Gamma \cap H_\theta$.

For all classes $\Gamma$, $\text{UR}_\alpha(\Gamma)$ implies $\text{RA}_\alpha(\Gamma)$.

Question

Is $\text{UR}_\omega(\Gamma)$ consistent? (If yes, we suspect $I_3$ to be needed)

Question

Can we obtain stronger forms of generic absoluteness from $\text{RA}_\alpha(\Gamma)$ or $\text{UR}_\alpha(\Gamma)$?
Thanks for your attention!
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