

Resurrection axioms and generic absoluteness

Giorgio Audrito
(joint work with Matteo Viale)

University of Torino

February 1, 2015

Outline

- 1 Generic absoluteness
 - Notation and conventions
 - Generic absoluteness in ZFC
- 2 Forcing axioms
 - Density properties
 - Absoluteness results
- 3 Resurrection axioms
 - Consistency strength of RA
 - RA as density property
 - Iterated resurrection axiom
- 4 Main results
 - Generic absoluteness from resurrection
 - Consistency strength
- 5 Stronger forms of resurrection

Hilbert's Program (finding a complete and consistent theory for mathematics) had an abrupt stop after Gödel's Incompleteness Theorem in 1931.

The question whether it would be possible to have **empiric completeness** was left open, even if early results showed that ZFC does not have such a behavior.

Many of this results are obtained using forcing, thus in order to achieve empiric completeness we need to "rule it out". How?

Definition

A theory T has *generic absoluteness* for a family Θ of first-order formulas and a definable class Γ of CBAs iff in all models of T the truth values of formulas in Θ cannot be changed in forcing extensions obtained by CBAs in Γ which preserves T .

We shall base on boolean valued models approach to forcing, and consider the following classes Γ of CBAs defined by properties interesting for forcing:

- Ω , the class of all CBAs,
- κ -distributive, κ -cc
- axiom- A , proper, semiproper (SP),
- stationary set preserving (SSP).

We shall equip a class Γ with two partial orders:

- $\mathbb{B} \leq_{\Gamma} \mathbb{C}$ iff there exists a complete homomorphism $i : \mathbb{C} \rightarrow \mathbb{B}$ such that the quotient algebra $\mathbb{B}/_{i[\dot{G}_{\mathbb{C}}]}$ is in Γ with boolean value $\mathbb{1}_{\mathbb{C}}$,
- $\mathbb{B} \leq_{\Gamma}^* \mathbb{C}$ iff there exists a complete *injective* homomorphism as above.

We denote by $\mathbb{U}_{\kappa}^{\Gamma}$ (*category forcing*) the set $\Gamma \cap H_{\kappa}$ ordered by \leq_{Γ} .

We say that Γ is iterable iff it is closed under two-step iterations, lottery sums and the order \leq_{Γ}^* is closed for set-sized descending sequences of elements of Γ . Most of the interesting classes are iterable, with the notable exception of SSP.

Definition

A theory T has *generic absoluteness* for a family Θ of first-order formulas and a definable class Γ of CBAs iff in all models of T the truth values of formulas in Θ cannot be changed in forcing extensions obtained by CBAs in Γ which preserves T .

Fundamental generic absoluteness results are known in the literature for ZFC with large cardinals, e.g:

- ZFC: generic absoluteness for $\Gamma = \Omega$ and $\Theta = \Sigma_2^1(\mathbb{R})$ (*Shönfield*)
- ZFC + \exists class many Woodin cardinals limit of Woodin cardinals:
generic absoluteness for $\Gamma = \Omega$ and Θ the formulas with real parameters relativized to $L(\mathbb{R})$ (*Woodin*)

More generic absoluteness results can be obtained in extensions of ZFC with forcing axioms.

Intuitively, forcing axioms postulate that V is closed under taking suitable forcing extensions over small set models of ZFC.

Definition

$\text{FA}(\Gamma)$ states that for all $\mathbb{B} \in \Gamma$ and collection \mathcal{D} of \aleph_1 -many dense subsets of \mathbb{B} , there exists a filter F meeting all of them.

Note that the same sentence for \aleph_0 -many dense subsets is Baire's Category Theorem.

Other variations we will consider are $\text{BFA}(\Gamma)$ (weakening) and $\text{FA}^{++}(\Gamma)$ (strengthening). Recall that MM , PFA are shorthands for $\text{FA}(\text{SSP})$, $\text{FA}(\text{proper})$.

Many of the commonly used forcing axiom can be restated as density properties (under suitable large cardinal hypothesis):

- BFA(Γ) holds iff the class $\{\mathbb{B} \in \Gamma : H_{\aleph_2} \prec_1 V^{\mathbb{B}}\}$ is dense in (Γ, \leq_{all}) ,
- FA(SSP) holds iff $\{\mathbb{B} \in SSP : \mathbb{B} \text{ is presaturated}\}$ is dense in (SSP, \leq_{all})
(class many Woodin cardinals),
- FA⁺⁺(SSP) holds iff the same class is dense in (SSP, \leq_{SSP})
(class many Woodin cardinals),
- MM⁺⁺⁺ holds iff $\{\mathbb{B} \in SSP : \mathbb{B} \text{ is strongly presaturated}\}$ is dense in (SSP, \leq_{SSP}) .

Examples of generic absoluteness results known in literature for extensions of ZFC with forcing axioms are:

- $\text{BFA}(\Gamma)$ is equivalent to ZFC having generic absoluteness for Θ the Σ_1 formulas with parameters relativized to H_{\aleph_2} and CBAs in Γ (*Bagaria*),
- $\text{ZFC} + \text{MM}^{+++} + \exists$ class many superhuge cardinals has generic absoluteness for $\Gamma = \text{SSP}$ and Θ the formulas relativized to $L([\text{ON}]^{\aleph_1})$ (*Viale*).

We show that strong generic absoluteness results can be obtained from resurrection axioms (of lower consistency strength).

We can develop the definition of the resurrection axiom starting from a model-theoretic point of view.

Theorem

Let $M \subset N$ be models of a language \mathcal{L} . Then TFAE:

- M is existentially closed in N ($M \prec_1 N$),
- M has **resurrection**, i.e. it exists a larger $M' \supseteq N$ such that $M \prec M'$

If we restrict the above properties to models of set theory of the form H_c^M where $c = \aleph_2$ and consider only model extensions obtained by forcing in a fixed class Γ , we obtain respectively:

- M satisfies $\text{BFA}(\Gamma)$,
- M satisfies $\text{RA}(\Gamma)$, the resurrection axiom

Resurrection axioms have been introduced recently by Hamkins and Johnstone, and are interesting since they can prove some consequences of FA, while having much lower consistency strength (for $\Gamma \neq \text{SSP}$).

In particular, we have that:

- $\text{RA}(\Gamma)$ for all mentioned Γ implies that $\mathfrak{c} \leq \aleph_2$,
- $\text{RA}(\Gamma) + \neg \text{CH}$ implies $\text{BFA}(\Gamma)$,
- $\text{FA}(\Gamma)$ is consistent relative to a supercompact cardinal (*Foreman, Magidor, Shelah*),
- $\text{RA}(\Gamma)$ for iterable Γ is consistent relative to a Mahlo cardinal (*Hamkins, Johnstone*),
- $\text{RA}(\text{SSP})$ is consistent relative to an inaccessible limit of Woodin cardinals above a supercompact cardinal (*Asperó*).

The resurrection axiom is conveniently stated as a density property:

Definition

$\text{RA}(\Gamma)$ holds iff the class $\left\{ \mathbb{B} \in \Gamma : H_c \prec H_c^{V^{\mathbb{B}}} \right\}$ is dense in (Γ, \leq_Γ) .

From $\text{RA}(\Gamma)$ we can already prove a weak form of generic absoluteness:

Theorem (Viale)

$\text{ZFC} + \text{RA}(\Gamma)$ has generic absoluteness for Θ the Σ_2 formulas with parameters relativized to H_c and forcing in Γ .

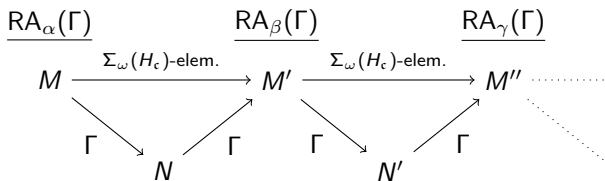
To achieve a stronger generic absoluteness result we need a stronger definition.

Definition (iterated resurrection axiom)

$\text{RA}_\omega(\Gamma)$ postulates that it is possible to resurrect the theory of H_c any fixed finite number of times.

Precisely, $\text{RA}_\alpha(\Gamma)$ is the assertion:

$\forall \beta < \alpha$ and $\forall N \supseteq M$ obtained by forcing in Γ ,
 $\exists M' \supseteq N$ a further extension by a forcing in Γ ,
 such that $H_c^M \prec H_c^{M'}$ and M' satisfies $\text{RA}_\beta(\Gamma)$.



$$\alpha > \beta > \gamma > \dots$$

Also the iterated resurrection axiom is conveniently stated as a density property:

Definition

$RA_\alpha(\Gamma)$ holds iff for all $\beta < \alpha$ the class

$$\left\{ \mathbb{B} \in \Gamma : H_c \prec H_c^{V^{\mathbb{B}}} \wedge V^{\mathbb{B}} \models RA_\beta(\Gamma) \right\}$$

is dense in (Γ, \leq_Γ) .

From this strengthened axiom we can obtain:

Theorem (A., Viale)

$ZFC + RA_\omega(\Gamma)$ has generic absoluteness for Θ the formulas relativized to H_c and forcing in Γ .

This directly improves the generic absoluteness result about $ZFC + RA(\Gamma)$, whereas with respect to Viale's absoluteness about $ZFC + MM^{+++} + LC$:

- Θ is smaller since $H_c \subset L([ON]^{\aleph_1})$,
- it is more general since it holds for any Γ (not only SSP),
- it has lower consistency strength

Lemma

$ZFC + \text{RA}_n(\Gamma)$ has generic absoluteness for Θ the Σ_{n+1} formulas relativized to H_c and forcing in Γ .

Proof.

By induction on n , consider a Σ_{n+1} formula $\phi = \exists x\psi(x)$ and draw the following:

$$\begin{array}{ccccc}
 \underline{\text{RA}_n(\Gamma)} & & H_c^M & \xrightarrow{\Sigma_\omega} & H_c^{M'} & & \underline{\text{RA}_{n-1}(\Gamma)} \\
 & & \searrow & & \nearrow & & \\
 & & \Sigma_n & & \Sigma_n & & \\
 & & & & H_c^N & & \\
 & & & & \underline{\text{RA}_n(\Gamma)} & &
 \end{array}$$

- $M \models \psi^{H_c}(a) \Rightarrow N \models \psi^{H_c}(a)$ so $M \models \exists x\psi^{H_c}(x) \Rightarrow N \models \exists x\psi^{H_c}(x)$,
- $N \models \exists x\psi^{H_c}(x) \Rightarrow M' \models \exists x\psi^{H_c}(x)$ (same argument) $\Rightarrow M \models \exists x\psi^{H_c}(x)$ (elementarity).

Theorem (A., Viale)

The following holds:

- $\text{RA}_{\text{ON}}(\Gamma)$ for iterable Γ is consistent relative to a Mahlo cardinal,
- $\text{RA}_{\text{ON}}(\text{SSP})$ is consistent relative to a stationary limit of supercompact cardinals,
- $\text{MM}^{+++} \Rightarrow \text{RA}_{\text{ON}}(\text{SSP})$.

Sketchy proof.

To prove consistency of $\text{RA}_\alpha(\Gamma)$ with Γ iterable (as for $\text{FA}(\Gamma)$ and variations), we use lottery iteration forcing with respect to suitable fast-growing (Menas) function $f : \kappa \rightarrow \kappa$ for a large enough cardinal κ .

$$\mathbb{B}_0 = 2$$

$$\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha * \dot{\mathbb{C}}_\alpha \text{ where } \dot{\mathbb{C}}_\alpha = \prod (\Gamma \cap H_{f(\alpha)}^{V^{\mathbb{B}_\alpha}})$$

\mathbb{B}_α for α limit is a lower bound in Γ for the chain $\langle \mathbb{B}_\beta : \beta < \alpha \rangle$

For $\Gamma = \text{SSP}$ we use the category forcing $\mathbb{U}_\kappa^{\text{SSP}}$ for a large enough cardinal κ . □

We developed a strengthening $RA_\alpha(\Gamma)$ of $RA(\Gamma)$. Other strengthened forms have been developed:

Definition (Hamkins, Johnstone)

The *boldface resurrection axiom* $\widetilde{RA}(\Gamma)$ holds iff for every $A \subseteq \mathfrak{c}$ the following class:

$$\{\mathbb{B} \in \Gamma : \exists A^* \subseteq \mathfrak{c}^{\mathbb{B}} \langle H_{\mathfrak{c}}, \in, A \rangle \prec \langle H_{\mathfrak{c}}^{\mathbb{B}}, \in, A^* \rangle\}$$

is dense in (Γ, \leq_{Γ}) .

Its consistency can be proved from a strongly uplifting cardinal (below subtle and 0^\sharp , above weakly compact).

For all classes Γ , $\widetilde{RA}(\Gamma)$ implies $RA(\Gamma)$.

It does not imply $RA_\omega(\Gamma)$.

We developed a strengthening $RA_\alpha(\Gamma)$ of $RA(\Gamma)$. Other strengthened forms have been developed:

Definition (Tsaprounis)

The *unbounded resurrection axiom* $UR(\Gamma)$ holds iff for every $\theta > \mathfrak{c}$ the class

$$\left\{ \mathbb{B} \in \Gamma : \exists j : H_\theta \rightarrow H_{j(\theta)}^{V^{\mathbb{B}}} \text{ crit}(j) = \mathfrak{c} \wedge j(\mathfrak{c}) > \theta \right\}$$

is dense in (Γ, \leq_Γ) below $\Gamma \cap H_\theta$, and $\mathfrak{c} = \omega_2$.

Its consistency can be proved from an extendible cardinal (higher consistency strength than $FA(\Gamma)$).

For most classes Γ , $UR(\Gamma)$ implies $FA^{++}(\Gamma)$ and $\widetilde{RA}(\Gamma)$.

It does not imply $RA_\omega(\Gamma)$.

It is also possible to combine the iteration idea of $RA_\alpha(\Gamma)$ with the boldface version $\widetilde{RA}(\Gamma)$.

Definition (A., Hamkins, Johnstone, Viale)

The *boldface α -resurrection axiom* $\widetilde{RA}_\alpha(\Gamma)$ holds iff for every $A \subseteq \mathfrak{c}$ and $\beta < \alpha$ the following class:

$$\left\{ \mathbb{B} \in \Gamma : \exists A^* \subseteq \mathfrak{c}^{\mathbb{B}} \langle H_{\mathfrak{c}}, \epsilon, A \rangle \prec \langle H_{\mathfrak{c}^{\mathbb{B}}}, \epsilon, A^* \rangle \wedge V^{\mathbb{B}} \models \widetilde{RA}_\beta(\Gamma) \right\}$$

is dense in (Γ, \leq_Γ) .

Its consistency can be proved from an α -strongly uplifting cardinal (like $\widetilde{RA}(\Gamma)$, below 0^\sharp and above weakly compact).

For all classes Γ , $\widetilde{RA}_\alpha(\Gamma)$ implies $RA_\alpha(\Gamma)$.

Definition

The *unbounded α -resurrection axiom* $\text{UR}_\alpha(\Gamma)$ holds iff for every $\theta > \mathfrak{c}$ and $\beta < \alpha$ the class

$$\left\{ \mathbb{B} \in \Gamma : \exists j : H_\theta \rightarrow H_{j(\theta)}^{V^\mathbb{B}} \text{ crit}(j) = \mathfrak{c} \wedge j(\mathfrak{c}) > \theta \wedge V^\mathbb{B} \models \text{UR}_\beta(\Gamma) \right\}$$

is dense in (Γ, \leq_Γ) below $\Gamma \cap H_\theta$.

For all classes Γ , $\text{UR}_\alpha(\Gamma)$ implies $\text{RA}_{\sim\alpha}(\Gamma)$.

Question





Is $\text{UR}_\omega(\Gamma)$ consistent? (If yes, we suspect I_3 to be needed)

Question





Can we obtain stronger forms of generic absoluteness from $\text{RA}_{\sim\alpha}(\Gamma)$ or $\text{UR}_\alpha(\Gamma)$?

Thanks for your attention!

Bibliography I

-  Giorgio Audrito, Joel David Hamkins, Thomas A. Johnstone, and Matteo Viale.
The boldface iterated resurrection axiom and the α -superstrongly unfoldable cardinals.
in preparation, 2015.
-  Giorgio Audrito and Matteo Viale.
Absoluteness via resurrection.
arXiv:1404.2111 (submitted to *Journal of Symbolic Logic*), 2014.
-  Joan Bagaria.
Bounded forcing axioms as principles of generic absoluteness.
Arch. Math. Logic, 39(6):393–401, 2000.
-  Joel David Hamkins and Thomas A. Johnstone.
Resurrection axioms and uplifting cardinals.
Arch. Math. Logic, 53(3-4):463–485, 2014.

Bibliography II

-  Joel David Hamkins and Thomas A. Johnstone.
Strongly uplifting cardinals and the boldface resurrection axioms.
[arXiv:1403.2788](#) (under review), 2014.
-  Itay Neeman and Jindřich Zapletal.
Proper forcings and absoluteness in $L(\mathbf{R})$.
Comment. Math. Univ. Carolin., 39(2):281–301, 1998.
-  Konstantinos Tsaprounis.
On resurrection axioms.
in preparation, 2013.
-  Matteo Viale.
Category forcings, MM^{+++} , and generic absoluteness for the theory of strong forcing axioms.
[arXiv:1305.2058](#), 2013.