Fragments of Martin's Axiom

and
the existence of a non-special Aronszajn tree

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Todorcevic's axioms on fragments of $MA_{\aleph_1}$

Definition (Martin-Solovay). $\mathsf{MA}_{\aleph_1}$: $\forall\, \mathbb{P}$ *ccc*
$\forall \{D_\alpha; \alpha \in \omega_1\}$ *dense subsets of* $\mathbb{P}$
$\exists\, G \subseteq \mathbb{P}$ *filter s.t.* $D_\alpha \cap G \neq \emptyset$ *for each* $\alpha \in \omega_1$.

Definition (Todorčević). $\mathcal{K}_{<\omega}$: *every ccc forcing* $\mathbb{P}$ *has precaliber* $\aleph_1$, *i.e.*

$\forall\, I \in [\mathbb{P}]^{\aleph_1}$
$\exists I' \in [I]^{\aleph_1}$ *such that any finite subset of* I' *has a common extension in* $\mathbb{P}$.

For each $n \in \omega$, $\mathcal{K}_n$: *every ccc forcing* $\mathbb{P}$ *has property* K_n, *i.e.*

$\forall\, I \in [\mathbb{P}]^{\aleph_1}$
$\exists I' \in [I]^{\aleph_1}$ n*-linked, i.e. any subset of* I' *of size* n *has a common extension in* $\mathbb{P}$.

$\mathcal{C}^2$: $\forall\, \mathbb{P}$ *ccc* $\forall\, \mathbb{Q}$ *ccc,* $\mathbb{P} \times \mathbb{Q}$ *also ccc.*

Todorcevic's axioms on fragments of $MA_{\aleph_1}$

Definition (Todorčević). *A partition* $K_0 \cup K_1 = [\omega_1]^{<\aleph_0}$ *(or* $[\omega_1]^n$*) is ccc if* $[\omega_1]^1 \subseteq K_0$ *(or ignore it when* $[\omega_1]^n$*) and the forcing* $\mathbb{P}_{K_0}$

$$\mathbb{P}_{K_0} := \text{ the set of finite } K_0\text{-homogeneous subsets of } \omega_1, \quad \leq_{\mathbb{P}_{K_0}} := \supseteq,$$

has the ccc.

$\mathcal{K}'_{<\omega}$: $\forall$ *ccc partition* $[\omega_1]^{<\aleph_0} = K_0 \cup K_1$
$\exists H \in [\omega_1]^{\aleph_1}$ *such that* $[H]^{<\aleph_0} \subseteq K_0$.

For each $n \in \omega$, $\mathcal{K}'_n$: $\forall$ *ccc partition* $[\omega_1]^n = K_0 \cup K_1$
$\exists H \in [\omega_1]^{\aleph_1}$ *such that* $[H]^n \subseteq K_0$.

Todorcevic's axioms on fragments of $MA_{\aleph_1}$

Theorem (Todorčević).

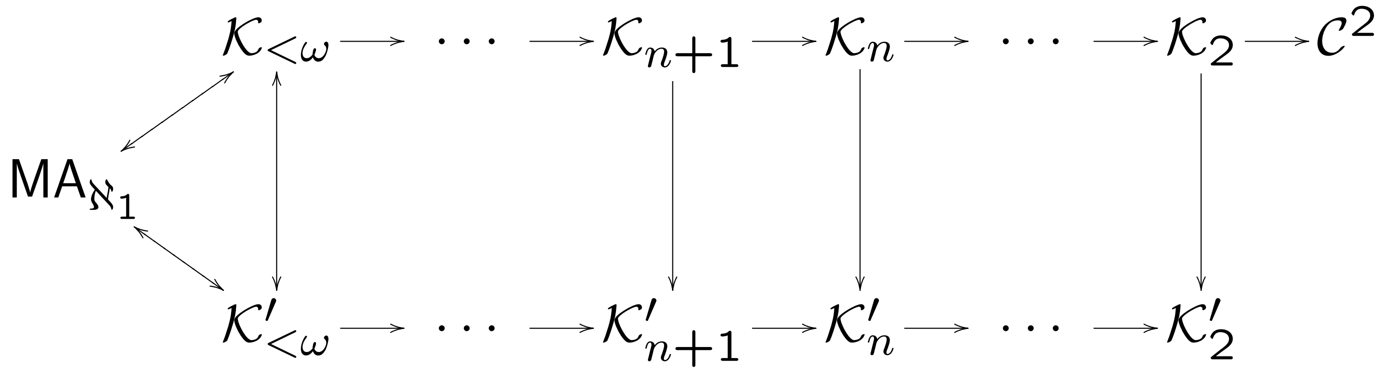
$\mathcal{C}^2 \Rightarrow$ *Suslin's Hypothesis,*
every (ω_1, ω_1)*-gap is indestructible,*
$\mathfrak{b} > \aleph_1$.

$\mathcal{K}_2 \Rightarrow \mathcal{K}'_2 \Rightarrow$ *every Aronszajn tree is special,*
every (ω_1, ω_1)*-gap is indestructible,*
$\mathfrak{b} > \aleph_1$.

$\mathcal{K}_3 \Rightarrow \mathcal{K}'_3 \Rightarrow$ $(2^{\omega_1}, <_{\text{lex}})$ *is embedded in* ω^ω/U *for every nontrivial* U*,*
$\text{add}(\mathcal{N}) > \aleph_1$.

$\mathcal{K}_4 \Rightarrow \mathcal{K}'_4 \Rightarrow$ *every ladder system on* ω_1 *can be uniformized,*
every uncountable set of reals is a Q*-set.*

Theorem (Todorčević-Veličković).



Todorcevic's axioms on fragments of $\mathsf{MA}_{\aleph_1}$

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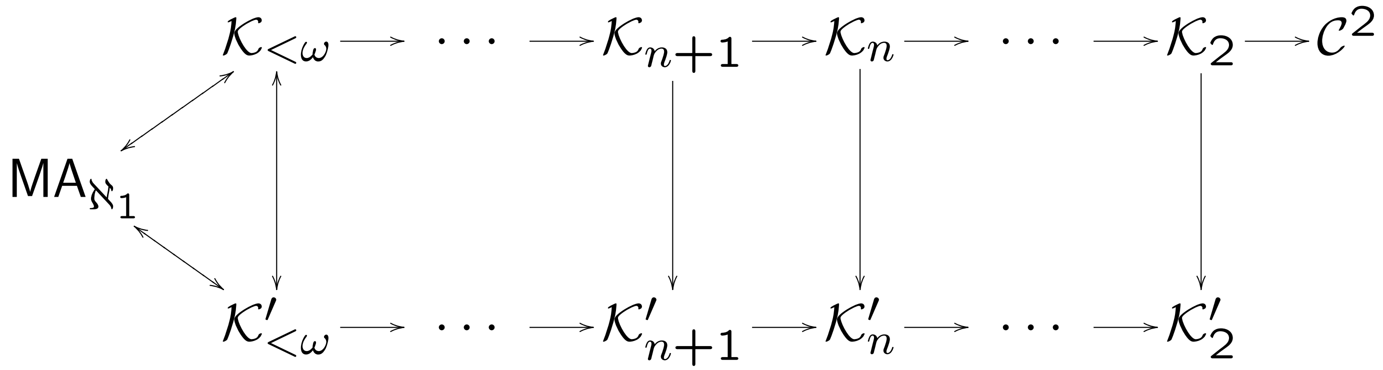
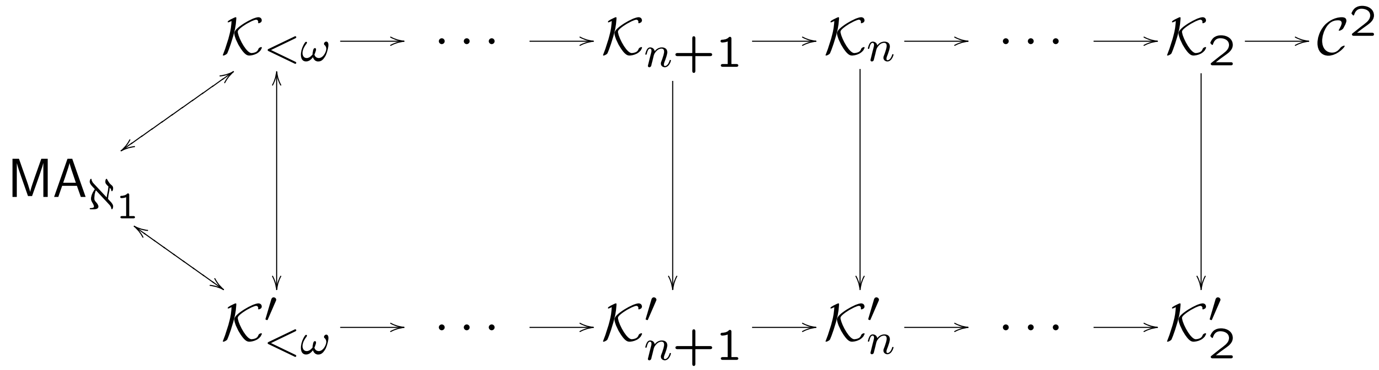


Question (Todorčević). *Are there other implications in the above diagram?*

Todorcevic's axioms on fragments of $\mathrm{MA}_{\aleph_1}$

Theorem (Todorčević-Veličković).



Question (Todorčević). *Are there other implications in the above diagram?*

Question. *For a subclass $\mathcal{P}$ of ccc forcings, what about the diagram:*

$$
\begin{array}{ccccccccc}
 & \mathcal{K}_{<\omega}(\mathcal{P}) & \cdots & \mathcal{K}_{n+1}(\mathcal{P}) & \mathcal{K}_n(\mathcal{P}) & \cdots & \mathcal{K}_2(\mathcal{P}) & \mathcal{C}^2(\mathcal{P}) \\
\mathrm{MA}_{\aleph_1}(\mathcal{P}) & & & & & & & \\
 & \mathcal{K}'_{<\omega}(\mathcal{P}) & \cdots & \mathcal{K}'_{n+1}(\mathcal{P}) & \mathcal{K}'_n(\mathcal{P}) & \cdots & \mathcal{K}'_2(\mathcal{P}) &
\end{array}
$$

The property $\mathsf{R}_{1,\aleph_1}$

Definition (Y.). *A partition* $[\omega_1]^2 = K_0 \cup K_1$ *has the property* $\mathsf{R}_{1,\aleph_1}$ *if for any large enough regular cardinal* κ*,*

$\forall$ *countable* $N \prec H(\kappa)$ *with* $K_0 \in N$
$\forall I \in [\omega_1]^{\aleph_1} \cap N$
$\forall \alpha \in \omega_1 \setminus N$
$\exists I' \in [I]^{\aleph_1} \cap N$ *such that* $\forall \beta \in I', \{\alpha, \beta\} \in K_0$.

Note that a partition on $[\omega_1]^2$ is ccc whenever it satisfies the property $\mathsf{R}_{1,\aleph_1}$.

Example. *For an Aronszajn tree* T*, define*

$$K_0 := \Big\{\{s,t\} \in [T]^2 : s \perp_T t\Big\}, \quad K_1 := [T]^2 \setminus K_0.$$

Then the partition $[T]^2 = K_0 \cup K_1$ *has the property* $\mathsf{R}_{1,\aleph_1}$.

Let countable $N \prec H(\aleph_2)$ *with* $T \in N$*,* $t \in T \setminus N$ *and* $I \in [T]^{\aleph_1} \cap N$*.*
Find $s_0, s_1 \in T \cap N$ *s.t. both* $\{u \in I : s_0 <_T u\}$ *and* $\{u \in I : s_1 <_T u\}$ *are uncountable.*
$\{u \in I : s_0 <_T u\}$ *or* $\{u \in I : s_1 <_T u\}$ *works well.*

The property $\mathsf{R}_{1,\aleph_1}$

Definition (Y.). *A partition* $[\omega_1]^2 = K_0 \cup K_1$ *has the property* $\mathsf{R}_{1,\aleph_1}$ *if for any large enough regular cardinal* κ,

$$\begin{aligned}
&\forall \text{ countable } N \prec H(\kappa) \text{ with } K_0 \in N \\
&\forall I \in [\omega_1]^{\aleph_1} \cap N \\
&\forall \alpha \in \omega_1 \setminus N \\
&\exists I' \in [I]^{\aleph_1} \cap N \text{ such that } \forall \beta \in I', \{\alpha, \beta\} \in K_0.
\end{aligned}$$

Example. $\mathcal{K}_2'(\mathsf{R}_{1,\aleph_1}) \Rightarrow$ *Suslin's Hypothesis,*
every (ω_1, ω_1)*-gap is indestructible,*
$\mathfrak{b} > \aleph_1$.

The property $\mathsf{R}_{1,\aleph_1}$

Definition (Y.). *A forcing notion* $\mathbb{P}$ *has the property* $\mathsf{R}_{1,\aleph_1}$ *if*

- $\mathbb{P} \subseteq [\omega_1]^{<\aleph_0}$ *uncountable and* $\leq_{\mathbb{P}} = \supseteq$, *and*
- *for any large enough regular cardinal* κ,

$$\begin{aligned}
&\forall \text{ countable } N \prec H(\kappa) \text{ with } \mathbb{P} \in N \\
&\forall I \in [\mathbb{P}]^{\aleph_1} \cap N \text{ which forms a } \Delta\text{-system with root } \nu \\
&\forall \sigma \in \mathbb{P} \setminus N \text{ with } \sigma \cap N = \nu \\
&\exists I' \in [I]^{\aleph_1} \cap N \text{ such that } \forall \tau \in I', \sigma \not\perp_{\mathbb{P}} \tau.
\end{aligned}$$

Example.

- *For any* $\mathsf{R}_{1,\aleph_1}$ *partition* $[\omega_1] = K_0 \cup K_1$, *the forcing* $\mathbb{P}_{K_0}$

$$\mathbb{P}_{K_0} := \text{ the set of finite } K_0\text{-homogeneous subsets of } \omega_1, \quad \leq_{\mathbb{P}_{K_0}} := \supseteq,$$

  satisfies the property $\mathsf{R}_{1,\aleph_1}$.

- $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1}) \Rightarrow \mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ *and every Aronszajn tree is special.*

The property $\mathsf{R}_{1,\aleph_1}$

Theorem (Shelah)**.** *It is consistent that there exists a non-special Aronszajn tree and Suslin's Hypothesis holds.*

Theorem (Y.)**.** *It is consistent that there exists a non-special Aronszajn tree and* $\mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ *holds.*

Therefore $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *and* $\mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ *are different.*

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Therefore $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$ *and* $\mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ *are different.*

Remember:
Theorem (Todorčević-Veličković)**.** $\mathsf{MA}_{\aleph_1} \Leftrightarrow \mathcal{K}_{<\omega}$.

Todorčević orderings

Definition (Todorčević, Balcar-Pazák-Thümmel). *For a topological space X, $\mathbb{T}(X)$ is the set of all subsets of X which are unions of finitely many convergent sequences including their limit points, and for each p and q in $\mathbb{T}(X)$, $q \leq_{\mathbb{T}(X)} p$ iff $q \supseteq p$ and $q^d \cap p = p^d$.*

Theorem (Todorčević).
- *$\mathbb{T}(\mathbb{R})$ is a non-σ-linked ccc forcing.*
- *If $\mathfrak{b} = \aleph_1$, $\mathbb{T}(\mathbb{R})$ doesn't have property K.*

Theorem (Balcar-Pazák-Thümmel). *It is consistent that there exists a topological space X such that $\mathbb{T}(X)$ is not ccc.*

Theorem (Thümmel). $\mathbb{T}\left(\left(\bigcup_{\alpha\in\omega_1} {}^{\alpha+1}(\omega^*), <_{\text{lex}}\right)\right)$ *satisfies the σ-finite cc, but doesn't satisfies the σ-bounded cc.*

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Theorem (Y.)**.** *It is consistent that there exists a non-special Aronszajn tree, $\mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ holds and $\mathcal{K}_{<\omega}\left(\left\{\mathbb{T}(X);\ \text{second countable } X\right)\right\})$ also holds.*

Appendices

Chodounský-Zapletal's Y-cc

Theorem (Y.). *For a topological space* X*, if* $\mathbb{T}(X)$ *satisfies the ccc, then* $\mathbb{T}(X)$ *adds no random reals.*

They develop this.

Chodounský-Zapletal's Y-cc

Definition (Chodounský-Zapletal). *A forcing $\mathbb{P}$ satisfies Y-cc if*

$\forall$ *countable* $M \prec H(\theta)$ *with* $\mathbb{P} \in M$
$\forall q \in \mathbb{P}$
$\exists F \in M$ *filter on* $\mathrm{ro}(\mathbb{P})$ *such that* $\left\{ r \in \mathrm{ro}(\mathbb{P}) \cap M ; q \leq_{\mathrm{ro}(\mathbb{P})} r \right\} \subseteq F$.

The followings are forcings which satisfies Y-cc:

- A σ-centered forcing satisfies Y-cc.
- For a partition $[X]^2 = K_0 \cup K_1$, define

$$\mathbb{P}_{K_0} := \text{ the set of finite } K_0\text{-homogeneous subsets of } X, \quad \leq_{\mathbb{P}_{K_0}} := \supseteq,$$

$$\mathbb{Q}_{K_0} := [X]^{<\aleph_0}, \quad q \leq_{\mathbb{Q}_{K_0}} p :\iff q \supseteq p \text{ and } \forall x \in q \setminus p \, \forall y \in p \Big(\{x, y\} \in K_0 \Big).$$

  If $\mathbb{Q}_{K_0}$ satisfies the ccc, then both $\mathbb{P}_{K_0}$ and $\mathbb{Q}_{K_0}$ satisfy Y-cc.

- For a topological space X, if $\mathbb{T}(X)$ satisfies the ccc, then $\mathbb{T}(X)$ satisfies Y-cc.

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The followings are forcings with Y-cc:

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  If $\mathbb{Q}_{K_0}$ satisfies the ccc, then both $\mathbb{P}_{K_0}$ and $\mathbb{Q}_{K_0}$ satisfy Y-cc.

- For a topological space X, if $\mathbb{T}(X)$ satisfies the ccc, then $\mathbb{T}(X)$ satisfies Y-cc.

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Proof. Let $\mathbb{P}$: Y-cc,
$\dot{x}$: ro($\mathbb{P}$)-name for a real in ${}^{\omega}2$,
$p \in \mathbb{P}$,
$M \prec H(\theta)$: countable with $\{\mathbb{P}, \dot{x}, p\} \in M$,
$\{U_n; n \in \omega\}$: open sets such that ${}^{\omega}2 \cap M \subseteq \bigcap_{n\in\omega} U_n$ measure zero.

Show that $p \Vdash_{\text{ro}(\mathbb{P})}$ " $\dot{x} \in \bigcap_{n\in\omega} U_n$ ".

Theorem (Chodounský-Zapletal)**.** *A Y-cc forcing adds no random reals.*

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Assume not, then we can take $q \leq_{\mathbb{P}} p$ and $m \in \omega$ such that $q \Vdash_{\mathrm{ro}(\mathbb{P})}$ " $\dot{x} \notin U_m$ ".

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By Y-cc of $\mathbb{P}$, there is a filter $F \in M$ on ro($\mathbb{P}$) with $\left\{r \in \text{ro}(\mathbb{P}) \cap M : q \leq_{\text{ro}(\mathbb{P})} r\right\} \subseteq F$.

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Define

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Because, if S is finite, then there exists $k \in \omega$ such that $S \subseteq 2^{<k}$, but then

$$0 \neq \prod_{v\in{}^{k}2} [\![\dot{x}{\restriction}k \neq v]\!]_{\text{ro}(\mathbb{P})} \Vdash_{\text{ro}(\mathbb{P})} \text{" } \dot{x}{\restriction}k \notin {}^{k}2 \text{"},$$

which is a contradiction.

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So we can take $u \in {}^{\omega}2 \cap M$ with $\forall k$, $u\restriction k \in S$, and take $l \in \omega$ with $[u\restriction l] \subseteq U_m$.

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Note that $S \in M$ and $(S, \subseteq)$ forms a tree. Point : S is infitite.
So we can take $u \in {}^{\omega}2 \cap M$ with $\forall k$, $u\restriction k \in S$, and take $l \in \omega$ with $[u\restriction l] \subseteq U_m$.
Then $q \cdot [\![\dot{x}\restriction l = u\restriction l]\!]_{\mathrm{ro}(\mathbb{P})} \neq 0$,

Theorem (Chodounský-Zapletal). *A Y-cc forcing adds no random reals.*

Proof. Let $\mathbb{P}$: Y-cc,
$\dot{x}$: ro($\mathbb{P}$)-name for a real in ${}^{\omega}2$,
$p \in \mathbb{P}$,
$M \prec H(\theta)$: countable with $\{\mathbb{P}, \dot{x}, p\} \in M$,
$\{U_n; n \in \omega\}$: open sets such that ${}^{\omega}2 \cap M \subseteq \bigcap_{n\in\omega} U_n$ measure zero.

Show that $p \Vdash_{\text{ro}(\mathbb{P})}$ " $\dot{x} \in \bigcap_{n\in\omega} U_n$ ".

Assume not, then we can take $q \leq_{\mathbb{P}} p$ and $m \in \omega$ such that $q \Vdash_{\text{ro}(\mathbb{P})}$ " $\dot{x} \notin U_m$ ".
By Y-cc of $\mathbb{P}$, there is a filter $F \in M$ on ro($\mathbb{P}$) with $\left\{r \in \text{ro}(\mathbb{P}) \cap M : q \leq_{\text{ro}(\mathbb{P})} r\right\} \subseteq F$.
Define

$$S := \left\{v \in 2^{<\omega}; [\![\dot{x}{\restriction}|v| \neq v]\!]_{\text{ro}(\mathbb{P})} \notin F\right\}.$$

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Then $q \cdot [\![\dot{x}{\restriction}l = u{\restriction}l]\!]_{\text{ro}(\mathbb{P})} \neq 0$,
Because, if $q \cdot [\![\dot{x}{\restriction}l = u{\restriction}l]\!]_{\text{ro}(\mathbb{P})} = 0$, then $q \cdot [\![\dot{x}{\restriction}l \neq u{\restriction}l]\!]_{\text{ro}(\mathbb{P})} = q$ holds, i.e.
$q \leq_{\text{ro}(\mathbb{P})} [\![\dot{x}{\restriction}l \neq u{\restriction}l]\!]_{\text{ro}(\mathbb{P})} \in \text{ro}(\mathbb{P}) \cap M$, which is a contradiction.

Theorem (Chodounský-Zapletal). *A Y-cc forcing adds no random reals.*

Proof. Let $\mathbb{P}$: Y-cc,
$\dot{x}$: ro($\mathbb{P}$)-name for a real in ${}^\omega 2$,
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So we can take $u \in {}^\omega 2 \cap M$ with $\forall k$, $u{\restriction}k \in S$, and take $l \in \omega$ with $[u{\restriction}l] \subseteq U_m$.
Then $q \cdot [\![\dot{x}{\restriction}l = u{\restriction}l]\!]_{\mathrm{ro}(\mathbb{P})} \neq 0$, and hence

$$q \cdot [\![\dot{x}{\restriction}l = u{\restriction}l]\!]_{\mathrm{ro}(\mathbb{P})} \Vdash_{\mathrm{ro}(\mathbb{P})} \text{“ } \dot{x} \in [\dot{x}{\restriction}l] = [u{\restriction}l] \subseteq U_m \text{“ },$$

which is a contradiction. □

The rectangle refining property

Definition (Larson–Todorčević). *A partition* $K_0 \cup K_1$ *on* $[\omega_1]^2$ *has the rectangle refining property if*

$$\forall I \in [\omega_1]^{\aleph_1} \; \forall J \in [\omega_1]^{\aleph_1}$$
$$\exists I' \in [I]^{\aleph_1} \; \exists J' \in [J]^{\aleph_1} \text{ such that } \forall \alpha \in I' \; \forall \beta \in J',\ \{\alpha, \beta\} \in K_0.$$

Definition (Y.). *A forcing notion* $\mathbb{P}$ *has the rectangle refining property if*

- $\mathbb{P} \subseteq [\omega_1]^{<\aleph_0}$ *uncountable and* $\leq_{\mathbb{P}} = \supseteq$, *and*
- $\forall I \in [\mathbb{P}]^{\aleph_1} \; \forall J \in [\mathbb{P}]^{\aleph_1}$, *if* $I \cup J$ *forms a* Δ*-system, then*
  $\exists I' \in [I]^{\aleph_1} \; \exists J' \in [J]^{\aleph_1}$ *such that* $\forall p \in I' \; \forall q \in J'$, $p \not\perp_{\mathbb{P}} q$.

Proposition.

$\mathcal{K}_2'$(rec) $\Rightarrow$ *Suslin's Hypothesis*
every (ω_1, ω_1)*-gap is indestructible,*
$\mathfrak{b} > \aleph_1$.

$\mathrm{MA}_{\aleph_1}$(rec $\cap$ FSCO$_2$) $\Rightarrow$ *every ladder system on* ω_1 *can be uniformized.*

The rectangle refining property

Theorem (Y.). *It is consistent that* $\mathsf{MA}_{\aleph_1}(\mathsf{rec})$ *holds and there exists an entangled set of reals, hence both* $\mathcal{C}^2$ *and* $\mathcal{K}_2'$ *fail.*

Theorem (Y.). $\mathcal{K}_2'(\mathsf{rec})$ *is equivalent to* $\mathcal{K}_2(\mathsf{rec})$*.*

Theorem (Y.). *It is consistent that* $\mathcal{K}_{<\omega}(\mathsf{rec} \cap \mathsf{FSCO}_2)$ *holds and* $\mathsf{MA}_{\aleph_1}(\mathsf{rec} \cap \mathsf{FSCO}_2)$ *fails.*

In particular, under $\mathsf{MA}_{\aleph_1}(S)$*,* S *forces* $\mathcal{K}_{<\omega}(\mathsf{rec} \cap \mathsf{FSCO}_2)$*.*

The rectangle refining property

Definition (Y.). FSCO$_2$ *is the collection of forcings* $\mathbb{P}$ *in* FSCO$_0$ *such that*

- *for any uncountable subset* I *of* $\mathbb{P}$, *there exists an uncountable subset* I' *of* I *such that for every finite subset* ρ *of* I', *if* ρ *has a common extension in* $\mathbb{P}$, $\bigcup \rho$ *is one of its common extensions, and*

- *for any uncountable subset* $\{\sigma_\alpha; \alpha \in \omega_1\}$ *of* $\mathbb{P}$, *there are an uncountable subset* Γ *of* ω_1 *and a sequence* $\langle \sigma'_\alpha; \alpha \in \Gamma \rangle$ *such that*
  - *for each* $\alpha \in \Gamma$, $\sigma'_\alpha \leq_{\mathbb{P}} \sigma_\alpha$ *(i.e.* $\sigma'_\alpha \supseteq \sigma_\alpha$*),*
  - *the set* $\{\sigma'_\alpha; \alpha \in \omega_1\}$ *forms a* Δ*-system, and*
  - *for every finite subset* ρ *of* Γ, *if the set* $\{\sigma'_\alpha; \alpha \in \rho\}$ *has a common extension in* $\mathbb{P}$, *then* $\bigcup_{\alpha \in \rho} \sigma'_\alpha$ *is its common extension and the set*

$$\left\{\beta \in \Gamma; \left\{\sigma'_\alpha; \alpha \in \rho\right\} \cup \left\{\sigma'_\beta\right\} \text{ has a common extension in } \mathbb{P}\right\}$$

    is uncountable.

Proposition. *If* $\mathbb{P} \in$ FSCO$_0$ *is ccc and closed under taking subsets, then* $\mathbb{P} \in$ FSCO$_2$.

Forcing extension with a separable measure algebra $\mathbb{B}$

Theorem (Roitman, 1979). *$\mathbb{B}$ forces the failure of $\mathcal{C}^2$.*

Theorem (Todorčević, 1986). *$\mathbb{B}$ adds an entangled set of reals, hence $\mathbb{B}$ forces the failure of $\mathcal{K}'_2$.*

So the forcing extension with $\mathbb{B}$ is not interesting from a veiwpoint of Todorcevic's question. But many people studies it.

Theorem (Laver, 1987). *Under* $\mathsf{MA}_{\aleph_1}$, *$\mathbb{B}$ forces every Aronszajn tree is special.*

Theorem. *Under* $\mathsf{MA}_{\aleph_1}$, *$\mathbb{B}$ forces the following statements:*

(Roitman? Kunen) $\mathsf{MA}_{\aleph_1}(\sigma\text{-linked})$,

(Hirschorm) *every (ω_1, ω_1)-gap is indestructible,*

(Moore) *every ladder system coloring can be uniformized,*

(Todorčević, Moore) *some statements about topology, e.g.* (S) *and* (L) *hold in the class of cometrizable spaces.*

Forcing with a non-separable measure algebra is quite different from forcing with a separable one.

For example, in the extension with a non-separable measure algebra,

(Moore) there exists a ladder system coloring which cannot be uniformized,

(Hirschorn) there exists a destructible gap.

Forcing extension with a separable measure algebra $\mathbb{B}$

Definition (Todorčević, Balcar–Pazák–Thümmel). *For a topological space X, $\mathbb{T}(X)$ is the set of all subsets of X which are unions of finitely many convergent sequences including their limit points, and for each p and q in $\mathbb{T}(X)$, $q \leq_{\mathbb{T}(X)} p$ iff $q \supseteq p$ and $q^d \cap p = p^d$.*

Theorem (Todorčević). • *$\mathbb{T}(\mathbb{R})$ is a non-σ-linked ccc forcing.*

- *if $\mathfrak{b} = \aleph_1$, $\mathbb{T}(\mathbb{R})$ doesn't have property K.*

Theorem (Balcar–Pazák–Thümmel). *It is consistent that there exists a topological space X such that $\mathbb{T}(X)$ is not ccc.*

Theorem (Thümmel). $\mathbb{T}\left(\left(\bigcup_{\alpha\in\omega_1} {}^{\alpha+1}(\omega^*), <_{\text{lex}}\right)\right)$ *satisfies the σ-finite cc, but doesn't satisfies the σ-bounded cc.*

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Theorem (Y.). *Under* $\mathrm{MA}_{\aleph_1}$, $\mathbb{B}$ *forces* $\mathrm{MA}_{\aleph_1}(\left\{\mathbb{T}(X); X \text{ second countable}\right\})$.

Forcing extension with a separable measure algebra $\mathbb{B}$

Theorem (Y.). *Under* $\mathsf{MA}_{\aleph_1}$, $\mathbb{B}$ *forces* $\mathsf{MA}_{\aleph_1}(\left\{\mathbb{T}(X); X \text{ second countable}\right\})$.

Sketch of a proof. Let $\dot{X}$ be a second countable space. For each $\varepsilon > 0$ $(\varepsilon < 1)$, define

$$\mathbb{P}_\varepsilon := \left\{\langle b, \dot{p}\rangle \,; b \in \mathbb{B},\ \mu(b) > \varepsilon,\ \dot{p} \text{ is a } \mathbb{B}\text{-name for a member of } \mathbb{T}(\dot{X})\right\},$$

$$\langle b, \dot{p}\rangle \leq_{\mathbb{P}_\varepsilon} \left\langle b', \dot{p}'\right\rangle :\iff b \leq_{\mathbb{B}} b' \text{ and } b \Vdash_{\mathbb{B}} \text{“}\, \dot{p} \leq_{\mathbb{T}(\dot{X})} \dot{p}' \,\text{”}.$$

It suffices to show that each $\mathbb{P}_\varepsilon$ is ccc.

Points of the proof are

- randomize the proof of the cccness of $\mathbb{T}(X)$ for a second countable X, and
- use an idea of Abraham–Rubin–Shelah's club method.

Interesting approach to Todorcevic's question

Question (Todorčević). *Under* $\mathsf{MA}_{\aleph_1}(S)$ *(or* $\mathsf{PFA}(S)$*), does* S *force* $\mathcal{C}^2$*?* $\mathcal{K}'_2$*?*

We note that a Suslin tree forces

- $\mathfrak{t} = \aleph_1$, so $\mathsf{MA}_{\aleph_1}(\sigma\text{-centered})$ fails,
- every ladder system has a coloring which cannot be uniformized, so $\mathcal{K}'_4$ fails,
- $\mathcal{K}'_3$ fails.

Question. *Under* $\mathsf{MA}_{\aleph_1}(S)$ *(or* $\mathsf{PFA}(S)$*), does* S *forces that there are no entangled set of reals?*
Or does a Suslin tree add an entangled set of reals?

Definition (Abraham–Rubin–Shelah). *A set E of reals is called entangled if E is uncountable and*

$\forall n \in \omega \quad \forall s \in {}^{n}\{0,1\} \quad \forall F \subseteq [E]^n$ *uncountable and pairwise disjoint*
$\exists x, y \in F$ *with* $x \neq y$ *such that*

$$\forall i < n \Big(x(i) < y(i) \iff s(i) = 0 \Big).$$

Suppose that $E = \{r_\alpha; \alpha \in \omega_1\}$ is an entangled set of reals, and define

$L := \Big\{ \Big\langle r_\alpha, r_{\alpha+1} \Big\rangle ; \alpha \in \omega_1 \text{ even} \Big\}$,

$\mathbb{P}_0 := \Big\{ p \in [L]^{<\aleph_0}; p \text{ is a chain in } L \Big\}$, $\leq_{\mathbb{P}_0} = \supseteq$,

$\mathbb{P}_1 := \Big\{ p \in [L]^{<\aleph_0}; p \text{ is an antichain in } L \Big\}$, $\leq_{\mathbb{P}_1} = \supseteq$.

Then both $\mathbb{P}_0$ and $\mathbb{P}_1$ are ccc and $\mathbb{P}_0 \times \mathbb{P}_1$ has an uncountable antichain.
$\mathbb{P}_0$ introduces a ccc partition which doesn't have uncountable 0-homogeneous sets.

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