The structure of Fréchet-Urysohn and radial spaces

Robert Leek
DPhil student, University of Oxford
robert.leek@maths.ox.ac.uk
www.maths.ox.ac.uk/people/profiles/robert.leek

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What are radial spaces?

**Definition**

$X$ is *Fréchet-Urysohn* at $x$ if whenever $x \in \overline{A}$, there exists a sequence $(x_n)_{n<\omega}$ in $A$ that converges to $x$.

If $X$ is Fréchet-Urysohn at every point $x$ in $X$, then we say that the space is *Fréchet-Urysohn*. 

**Fréchet-Urysohn space**

$(x_n)_{n<\omega} \to x$
What are radial spaces?

Definition

$X$ is radial at $x$ if whenever $x \in \overline{A}$, there exists a transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ in $A$ that converges to $x$.

If $X$ is radial at every point $x$ in $X$, then we say that the space is radial.
Definition

$X$ is \textit{first-countable} at $x$ if there exists a countable neighbourhood base for $x$. Equivalently, there exists a descending neighbourhood base $(U_n)_{n<\omega}$ for $x$. If $X$ is first-countable at every point $x$ in $X$, then we say that the space is \textit{first-countable}.

First countable space
Definition

$X$ is \textit{well-based} at $x$ if it has a well-ordered neighbourhood base with respect to $\supseteq$. If $X$ is well-based at every point $x$ in $X$, then we say that the space is \textit{well-based}.
### Some examples

**Definition**

- **LOTS** := Linearly-Ordered Topological Space
- **GO-space** := Generalised-Ordered space = Subspaces of **LOTS**

![Diagram of LOTS (or GO-space)](image-url)
Some examples

Definition

$\text{LOTS} := \text{Linearly-Ordered Topological Space}$

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Definition

**LOTS** := Linearly-Ordered Topological Space

**GO-space** := Generalised-Ordered space = Subspaces of **LOTS**

Definition

A *spoke* for a point is a well-based subspace containing that point.
Introducing spoke systems

Definition

A collection of spokes $\mathcal{S}$ for a point $x$ is a *spoke system* for $x$ if

$$
\mathcal{B} := \left\{ \bigcup_{S \in \mathcal{S}} B_S : \forall S \in \mathcal{S}, B_S \in \mathcal{N}_x^S \right\}
$$

is a neighbourhood base for $x$, where $\mathcal{N}_x^S$ is the collection of $S$-neighbourhoods of $x$, for each $S \in \mathcal{S}$. 

Theorem

Every point with a spoke system is radial.
Introducing spoke systems

**Definition**

A collection of spokes $\mathcal{I}$ for a point $x$ is a *spoke system* for $x$ if

$$
\mathcal{B} := \left\{ \bigcup_{S \in \mathcal{I}} B_S : \forall S \in \mathcal{I}, B_S \in \mathcal{N}_x^S \right\}
$$

is a neighbourhood base for $x$, where $\mathcal{N}_x^S$ is the collection of $S$-neighbourhoods of $x$, for each $S \in \mathcal{I}$. Equivalently, for all $A \subseteq X$ with $x \in \overline{A}$, there exists an $S \in \mathcal{I}$ such that $x \in A \cap \overline{S}$. 
Introducing spoke systems

**Definition**

A collection of spokes $\mathcal{I}$ for a point $x$ is a *spoke system* for $x$ if

$$\mathcal{B} := \left\{ \bigcup_{S \in \mathcal{I}} B_S : \forall S \in \mathcal{I}, B_S \in \mathcal{N}^S_x \right\}$$

is a neighbourhood base for $x$, where $\mathcal{N}^S_x$ is the collection of $S$-neighbourhoods of $x$, for each $S \in \mathcal{I}$. Equivalently, for all $A \subseteq X$ with $x \in \overline{A}$, there exists an $S \in \mathcal{I}$ such that $x \in \overline{A \cap S}$.

**Theorem**

*Every point with a spoke system is radial.*
Introducing spoke systems

**Definition**

A transfinite sequence \((x_\alpha)_{\alpha<\lambda}\) converges strictly to a point \(x\) if it converges to \(x\) and \(x\) is not in the closure of any initial segment; that is, \(x \notin \{x_\alpha : \alpha < \beta\}\), for all \(\beta < \lambda\).

**Lemma**

*If \(X\) is radial at \(x\) and \(x \in \overline{A}\), then there exists an injective transfinite sequence in \(A\) that converges strictly to \(x\).*
Introducing spoke systems

Definition

A transfinite sequence \((x_\alpha)_{\alpha<\lambda}\) \textit{converges strictly} to a point \(x\) if it converges to \(x\) and \(x\) is not in the closure of any initial segment; that is, \(x \notin \{x_\alpha : \alpha < \beta\}\), for all \(\beta < \lambda\).

Lemma

If \(X\) is radial at \(x\) and \(x \in \overline{A}\), then there exists an injective transfinite sequence in \(A\) that converges strictly to \(x\).

Lemma

Let \((x_\alpha)_{\alpha<\lambda}\) be an injective transfinite sequence that converges strictly to \(x\). Then \(S_{(x_\alpha)_{\alpha<\lambda}} := \{x\} \cup \{x_\alpha : \alpha < \lambda\}\) is a spoke for \(x\).
An internal characterisation of radiality

**Theorem**

For a point \( x \) in a topological space \( X \), the following are equivalent:

1. \( X \) is radial at \( x \).
2. \( X \) has an almost-independent spoke system \( \mathcal{S} \) at \( x \); that is, for distinct \( S, T \in \mathcal{S}, x \notin (S \cap T) \setminus \{x\} \).
An internal characterisation of radiality

**Theorem**

For a point $x$ in a topological space $X$, the following are equivalent:

1. $X$ is radial at $x$.
2. $X$ has an almost-independent spoke system $\mathcal{S}$ at $x$; that is, for distinct $S, T \in \mathcal{S}, x \notin (S \cap T) \setminus \{x\}$.

**Proof.**

If $X$ is radial at $x$ and not isolated, define:

$$
\mathcal{T} := \{f : \lambda \to X \setminus \{x\} \mid \lambda \leq |X|, f \text{ is injective and } f \to x \text{ strictly}\}
$$

$$
\mathcal{A} := \{\mathcal{T} \subseteq \mathcal{T} : \forall f, g \in \mathcal{T} \text{ distinct, } f^{-1}[\text{ran}(g)] \text{ is bdd. in } \text{dom}(f)\}
$$

Pick $\mathcal{T} \in \mathcal{A}$ maximal and define $\mathcal{I} := \{S_f : f \in \mathcal{T}\}$. □
Proposition

Let $\mathcal{S}$ be a spoke system for $x$ and $(x_\alpha)_{\alpha<\lambda}$ be a transfinite sequence clustering at $x$ with $x \notin \{x_\alpha : \alpha < \beta\}$ for $\beta < \lambda$, where $\lambda$ is a regular ordinal. Then there exists an $S \in \mathcal{S}$ and a subsequence of $(x_\alpha)_{\alpha<\lambda}$ contained in $S$ and converging to $x$. 
Some properties of spoke systems

Proposition

Let $\mathcal{S}$ be a spoke system for $x$ and $(x_\alpha)_{\alpha<\lambda}$ be a transfinite sequence clustering at $x$ with $x \notin \{x_\alpha : \alpha < \beta\}$ for $\beta < \lambda$, where $\lambda$ is a regular ordinal. Then there exists an $S \in \mathcal{S}$ and a subsequence of $(x_\alpha)_{\alpha<\lambda}$ contained in $S$ and converging to $x$.

Proof.

As $x \in \{x_\alpha : \alpha < \lambda\}$, there exists an $S \in \mathcal{S}$ such that $x \in \{x_\alpha : \alpha < \lambda\} \cap S$. Then $\chi(x, S) = \lambda$...
Some properties of spoke systems

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As $x \in \{x_\alpha : \alpha < \lambda\}$, there exists an $S \in \mathcal{S}$ such that $x \in \{x_\alpha : \alpha < \lambda\} \cap S$. Then $\chi(x, S) = \lambda\ldots$ ⌊

Proposition

If $\mathcal{S}$ is an independent spoke system for $x$ and $(x_\alpha)_{\alpha<\lambda} \subseteq X\setminus\{x\}$ converges to $x$, with $\lambda$ regular, then there exists $\mathcal{T} \in [\mathcal{S}]^{<\lambda}$ and $\beta < \lambda$ such that $\{x_\alpha : \alpha \in [\beta, \lambda)\} \subseteq \bigcup \mathcal{T}$. 
A point \( x \) in a space \( X \) is **strongly Fréchet** if for every decreasing sequence of subsets \( (A_n) \) with \( x \in \bigcap_{n \in \omega} \overline{A_n} \), there exists a sequence \( (x_n) \) converging to \( x \) with \( x_n \in A_n \) for all \( n \in \omega \).
Definition (Strongly Fréchet)

A point $x$ in a space $X$ is strongly Fréchet if for every decreasing sequence of subsets $(A_n)$ with $x \in \bigcap_{n \in \omega} A_n$, there exists a sequence $(x_n)$ converging to $x$ with $x_n \in A_n$ for all $n \in \omega$.

Theorem

Let $x$ be a Fréchet-Urysohn point in $X$. Then $x$ is strongly Fréchet if and only if for all (non-trivial) spoke systems $\mathcal{S}$ at $x$ and every countably infinite subset $A \subseteq \mathcal{S}$, there exists an $S \in \mathcal{S}$ such that $A \cap S \neq \{x\}$ for all $A \in \mathcal{A}$.

Sketch proof.

$\Rightarrow$: consider $A_n := \bigcup_{m=n}^{\infty} S_m$, where $(S_m) \subseteq \mathcal{S}$.

$\Leftarrow$: use Zorn’s Lemma (similar to proof of existence of almost-independent spoke systems).
### Definition (Independently-based)

We say that a point $x$ is *independently-based* if it has an independent spoke system $\mathcal{S}$; that is, $S \cap T = \{x\}$ for all distinct $S, T \in \mathcal{S}$. 

Equivalently, there exists a collection $C$ of nests of neighbourhoods of $x$ such that

$\{ \bigcap C : \forall C \in C, U_C \in C \}$ is a neighbourhood base for $x$ and for every selection $(U_C)_{C \in C}$, $\bigcap C \in C \cup C : (U_C \cap S_C)$

where $S_C : \bigcap \{ \bigcap D : D \in C, D \neq C \}$. 
Definition (Independently-based)

We say that a point $x$ is \textit{independently-based} if it has an \textit{independent} spoke system $\mathcal{S}$; that is, $S \cap T = \{x\}$ for all distinct $S, T \in \mathcal{S}$. Equivalently, there exists a collection $\mathcal{C}$ of nests of neighbourhoods of $x$ such that

$$\left\{ \bigcap_{C \in \mathcal{C}} U_C : \forall C \in \mathcal{C}, U_C \in \mathcal{C} \right\}$$

is a neighbourhood base for $x$ and for every selection $(U_C)_{C \in \mathcal{C}}$,

$$\bigcap_{C \in \mathcal{C}} U_C = \bigcup_{C \in \mathcal{C}} (U_C \cap S_C)$$

where $S_C := \bigcap\{\bigcap D : D \in \mathcal{C}, D \neq C\}$. 
\[ C_1 \in \mathcal{C}_1 \]

\[ C_2 \in \mathcal{C}_2 \]

\[ C_3 \in \mathcal{C}_3 \]
$C_1 \in \mathcal{C}_1$

$C_2 \in \mathcal{C}_2$

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$C_1 \cap C_2 \cap C_3$
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$C_1 \cap C_2 \cap C_3$
Theorem

A point $x$ in a space $X$ is first countable if and only if it is independently-based and strongly Fréchet.

Corollary

There exists a Fréchet-Urysohn space that is not independently-based.

Proof.

Take $X = \alpha D (\aleph_1)$. 
Theorem

A point \( x \) in a space \( X \) is first countable if and only if it is independently-based and strongly Fréchet.

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There exists a Fréchet-Urysohn space that is not independently-based.

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Take \( X = \alpha D(\aleph_1) \).  \( \square \)
Theorem

A point $x$ in a space $X$ is first countable if and only if it is independently-based and strongly Fréchet.

Corollary

There exists a Fréchet-Urysohn space that is not independently-based.

Proof.

Take $X = \alpha D(\aleph_1)$.

Theorem

There exists a Fréchet-Urysohn space with a point that is neither strongly Fréchet nor independently-based.
**Lemma (Reflection Lemma)**

Let $x$ be a Fréchet-Urysohn point, $\mathcal{I}, \mathcal{T}$ be spoke systems at $x$, with $\mathcal{T}$ independent. Then for all $K_S := \{ T \in \mathcal{T} : x \in (S \cap T) \setminus \{x\} \}$ is finite, for all $S \in \mathcal{I}$. 
Independently-based spaces

Lemma (Reflection Lemma)

Let \( x \) be a Fréchet-Urysohn point, \( \mathcal{S}, \mathcal{T} \) be spoke systems at \( x \), with \( \mathcal{T} \) independent. Then for all \( K_S := \{ T \in \mathcal{T} : x \in (S \cap T)\setminus\{x\} \} \) is finite, for all \( S \in \mathcal{S} \).

Sketch proof of previous theorem.

For \( x \in \mathbb{R}^2\setminus\{0\} \), define \( S_x := \{ y \in \mathbb{R}^2 : \|y - x\| = \|x\| \} \) and let \( \mathcal{B} := \{ \bigcup_{x \in \mathbb{R}^2\setminus\{0\}} (S_x \cap B(0, \epsilon_x)) : \forall x \in \mathbb{R}^2\setminus\{0\}, \epsilon_x > 0 \} \). If \( 0 \) is independently-based, then for each \( x \in \mathbb{R}^2\setminus\{0\} \), there exists an \( \epsilon_x > 0 \) such that \( S_x \cap S_y \cap B(0, \min(\epsilon_x, \epsilon_y)) = \{0\} \) for all distinct \( x, y \in \mathbb{R}^2\setminus\{0\} \). By the Baire category theorem, we obtain a contradiction.
Some problems

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<th>Proposition</th>
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<td>If $x$ is a Fréchet-Urysohn, non-first-countable point with a countable, almost-independent spoke system, then $\chi(x, X) = 0$.</td>
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Some problems

Proposition

If $x$ is a Fréchet-Urysohn, non-first-countable point with a countable, almost-independent spoke system, then $\chi(x, X) = 0$.

Question

What if $x$ has no countable, almost-independent spoke system?
Proposition

If \( x \) is a Fréchet-Urysohn, non-first-countable point with a countable, almost-independent spoke system, then \( \chi(x, X) = \emptyset \).

Question

What if \( x \) has no countable, almost-independent spoke system?

Question

Let \( \mathcal{A} \) be an almost-disjoint family on \( \omega \) and topologise \( \omega \cup \{\star\} \) by declaring \( A \) to be a sequence converging to \( \star \) and \( \{A \cup \{\star\} : A \in \mathcal{A}\} \) is a spoke system at \( \star \) (so \( \omega \cup \{\star\} \cong \Psi(\mathcal{A})/\mathcal{A} \)). What is the character of \( \star \)?
R. Leek.
Convergence properties and compactifications.
Submitted, 2014.

R. Leek.
An internal characterisation of radiality.