Mid point free sets

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Definition

- Let $(G, +)$ be any abelian Polish group. Let $c \in G$. We write $nc = c + \cdots + c$ if there are n terms in the sum. We say that $c \in G$ is a *midpoint* of $a, b \in G$ if $a \neq b$ and $2c = a + b$.
- We say that a subset A of an abelian froup G is *midpoint free* if no point of A is a midpoint of two other points of A.
- A subset A of a vector space V is called *rationally convex* if $q_1 v_1 + \ldots q_n v_n \in A$ for any finite sequence $v_1, \ldots, v_n$ of pairwise different elements from A and any sequence of positive rational numbers $q_1, \ldots, q_n$, such that $q_1 + \ldots + q_n = 1$.

Forbidden zones

- $Z_1(A) = \{v \in G : \ (\exists a, b \in A) \ a + b = 2v\}$
- $Z_2(A) = \{v \in G : \ (\exists a, b \in A) \ v = 2b - a\}$

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Let us observe that every maximal midpoint free set is not linearly independent over field $\mathbb{Q}$ of all rational numbers. Moreover, every maximal midpoint free set contains a Hamel basis.

Theorem (Erdös, Kakutani)

CH is equivalent to the existence of a partition of the real line into a countable family of independent sets with respect to the rationals $\mathbb{Q}$.

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Theorem

Real line can be decomposed into countably many rationally convex free sets.

Proof

Let $\{x_\xi : \xi < 2^\omega\}$ be a Hamel base of $\mathbb{R}$. For every sequence $(q_1, q_2, \dots, q_n)$ of rational numbers which are not equal to zero set

$$A(q_1, q_2, \dots q_n) = \{q_1 x_{\xi_1} + q_2 x_{\xi_2} + \dots + q_n x_{\xi_n} : \xi_1 < \xi_2 < \dots \xi_n\}.$$

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Let $X \leq \mathbb{R}$ be a proper countable linear subspace of $\mathbb{R}$ over rationals and $Z = span_{\mathbb{Q}}\{X \cup \{h\}\}$ for a some $h \in \mathbb{R} \setminus X$. Assume that $X = \bigcup_{n\in\omega} Q_n$ be a partition of X onto maximal midpoint free sets. Then there exists a docomposition of $Z = \bigcup_{n\in\omega} R_n$ onto maximal midpoint free set such that for any $n \in \omega$ we have $Q_n \subseteq R_n$.

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Theorem

CH implies the countable decomposition of the real line onto maximal midpoint free sets.

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Real line can be partitioned into continuum many maximal midpoint free sets.

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Proof

Let $\mathbb{Q} = \bigcup_{n\in\omega} Q_n$ be decomposition from Lemma. Assume that $0 \in Q_0$. Let $\{v_\alpha : \alpha \in 2^\omega\}$ be Hamel basis of $\mathbb{R}$. Let $f : 2^\omega \to \omega$ be such that $supp(f) = \{\alpha : f(\alpha) \neq 0\}$ be finite. Set

$$R_f = \left\{ \sum_{\alpha\in 2^\omega} q_\alpha v_\alpha : \forall\alpha \; q_\alpha \in Q_{f(\alpha)} \text{ and } q_\alpha = 0 \right.$$

$$\text{for all but finetely many } \alpha\text{'s}\}$$

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Every midpoint free set in a Euclidean space which has the Baire property is meager and every measurable midpoint free set has Lebesgue zero.

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Theorem

Every abelian Polish group such that the set $\{x \in G : \ x + x = a\}$ is countable for every $a \in G$ contains a midpoint free set which is also a Bernstein set.

Theorem

- Under CH there is a midpoint free set which is a Lusin set;
- It is relatively consistent with ZFC that $\neg CH$ and there is a midpoint free set which is also a Lusin set.

Of course, the same is true when we replace Lusin set by Sierpiński set.

Definition

$A \subseteq \mathbb{Z}^\omega$ is

- *bounded* if there is $f \in \mathbb{Z}^\omega$ such that $\forall a \in A\, \forall^\infty n\, |a(n)| \leq |f(n)|$;
- *unbounded* if A is not bounded;
- *dominating* if $\forall f \in \mathbb{Z}^\omega\, \exists a \in A\, \forall^\infty n\, |f(n)| \leq |a(n)|$.

Theorem

1. If $A \subseteq \mathbb{Z}^\omega$ is maximal midpoint free then A is unbounded.
2. There exists a maximal midpoint free $A \subseteq \mathbb{Z}^\omega$ which is dominating.
3. There exists a maximal midpoint free $A \subseteq \mathbb{Z}^\omega$ which is not dominating.

Example

The hyperbola $H = \{(x, y) \in \mathbb{R}^2 : \ xy = 1\}$ is a maximal closed midpoint free subset of the plane $\mathbb{R}^2$.

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$\mathbb{T} = S^1 \times S^1$, where addition is defined by the following formula:

$$(a, b) + (c, d) = (a + c \mod 1, b + d \mod 1).$$

A circle $C = \{(x, y) \in \mathbb{T} : \; (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{18}\}$ is a closed maximal midpoint free subset of $\mathbb{T}$.

Theorem

The set $F = \{a = .a_1 a_2 \ldots : a_i = 0 \text{ or } a_i = 1, i \in \mathbb{N}\}$ of those points in S^1 whose quaternary expansions have only digits 0 and 1 is a closed maximal midpoint free subset of S^1.

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Theorem

If E is a closed maximal midpoint free subset of S^1, then the first forbidden zone for E, $Z_1(E) = \{x : 2x = a + b : b, a \in E, a \neq b\}$, is a proper subset of E^c.

Theorem

There exists a maximal midpoint free subset F of the group $(\mathbb{Z}, +)$ with the first forbidden zone $Z_1(F)$ equal to the whole F^c.

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Let C_1 be the set of those points from the interval $[0, 1]$ whoce quartenary expansion contains only numbers: $0, 1$, and let C_2 be the set of those points from the interval $[0, 1]$ whose decimal expansion contains only numbers: $0, 2$. These sets are midpoint free and $C_1 + C_2$ is equal to the interval $[0, 1]$.

Theorem

There exists a maximal closed midpoint free subset F of the group $(\mathbb{R}, +)$ with the first forbidden zone $Z_1(F)$ equal to the whole F^c.

Thank you for your attention!