Compactifications of $\omega$ and the Banach space $c_0$

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Winter School in Abstract Analysis
Hejnice, February 2015
Banach spaces

Notation

$C(K)$ is the space of continuous functions $K \to \mathbb{R}$.

$c_0$ is the space of sequences $x = (x_n)_{n \in \omega}$ converging to 0.

$l_\infty$ is the space of bounded sequences, $l_\infty = C(\beta\omega)$.

Complemented subspaces

A closed subspace $Y$ of a Banach space $X$ is complemented if $X = Y \oplus Z$ for some closed subspace $Z \subseteq X$.

Equivalently, there is a bounded linear operator $P : X \to X$, which is a projection i.e. $P \circ P = P$, and such that $P(X) = Y$.

Classical results

(a) Sobczyk: If $X$ is separable then every isomorphic copy of $c_0$ in $X$ is complemented.

(b) Phillips: $c_0$ is not complemented in $l_\infty$. 
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$c_0$ in $C(K)$, $K$ infinite compact

$\mathcal{X} = \{ X \subseteq C(K) : X$ is isomorphic to $c_0 \}$;
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- $\mathcal{X}_c = \mathcal{X}$; examples: compact lines (Correa & Tausk).
Let $\gamma \omega$ be a compactification of $\omega$ so $\gamma \omega$ is compact and contains $\omega$ as a dense subset of isolated points. Then $C(\gamma \omega)$ contains (a natural copy of) $c_0$, namely $c_0 = \{ g \in C(\gamma \omega) : g|_{\gamma \omega \setminus \omega} \equiv 0 \}$.

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Problem Characterize $\gamma \omega$ such that $c_0$ is complemented in $C(\gamma \omega)$.

Recall that $c_0$ is complemented in $C(\gamma \omega)$ whenever $\gamma \omega$ is metrizable. $c_0$ is not complemented in $C(\beta \omega)$. 
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Compactifications of $\omega$ and subalgebras of $P(\omega)$

Every zerodimensional $\gamma\omega$ may be seen as the Stone space $\text{ult}(A)$ of some algebra $A \subseteq P(\omega)$ containing $\text{fin}$. We shall write $K_A = \text{ult}(A)$ for such a compactification and $K^*_A = K_A \setminus \omega$ for its remainder.

Finitely additive measures $\text{ba}^+(A)$ denote the space of all bounded finitely additive measures on $A$.

$\text{ba}^+(A) = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \text{ba}^+(A)\}$ is the space of all signed measures. Essentially, $\text{ba}^+(A)$ is the dual Banach space of all functionals on $C(K_A)$. 
Every zerodimensional $\gamma\omega$ may be seen as the Stone space $\text{ult}(\mathcal{A})$ of some algebra $\mathcal{A} \subseteq P(\omega)$ containing $\text{fin}$. We shall write $K_{\mathcal{A}} = \text{ult}(\mathcal{A})$ for such a compactification and $K_{\mathcal{A}}^* = K_{\mathcal{A}} \setminus \omega$ for its remainder.
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Finitely additive measures

Let $ba_+(\mathcal{A})$ denote the space of all bounded finitely additive measures on $\mathcal{A}$.

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is the space of all signed measures. Essentially, $ba(\mathcal{A})$ is the dual Banach space of all functionals on $C(K_{\mathcal{A}})$. 
Basic lemma

The following are equivalent for \( \text{fin} \subseteq A \subseteq \mathcal{P}(\omega) \):

(i) \( c_0 \) is complemented in \( C(KA) \);

(ii) there is a uniformly bounded sequence \((\nu_n)\) in \( \text{ba}(A) \) such that every \( \nu_n \) vanishes on \( \text{fin} \) and \( \nu_n - \delta_n \to 0 \).

Remarks

\( \nu_n - \delta_n \to 0 \) means that for every \( A \in A \)

\[ \lim_{n \to A} \nu_n(A) = 1 \text{ and } \lim_{n \to A} \nu_n(\omega \setminus A) = 0. \]

If \( c_0 \) is complemented in \( C(KA) \) then \( KA^* \) must carry a strictly positive measure.
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Lemma

The following are equivalent for $\text{fin} \subseteq \mathcal{A} \subseteq \mathcal{P}(\omega)$

(i) $c_0$ is complemented in $C(K_{2\mathbb{I}})$;

(ii) there is a uniformly bounded sequence $(\nu_n)_n$ in $\text{ba}(\mathcal{A})$ such that every $\nu_n$ vanishes on $\text{fin}$ and $\nu_n - \delta_n \to 0$.

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- If $c_0$ is complemented in $C(K_{2\mathbb{I}})$ then $K_{2\mathbb{I}}^*$ must carry a strictly positive measure.
Suppose that $\mathcal{F} \subseteq \mathcal{A} \subseteq \mathcal{P}(\omega)$ and the quotient map

$$\mathcal{A} \rightarrow \mathcal{A}/\mathcal{F},$$

admits a lifting. Then $c_0$ is complemented in $C(\mathcal{K}\mathcal{A})$.

Proof. By our assumption there is a homomorphism $\theta: \mathcal{A}/\mathcal{F} \rightarrow \mathcal{A}$, such that $\theta(a) \cdot = a$ for $a \in \mathcal{A}/\mathcal{F}$.

Define $\nu_n \in \text{ba}^+(\mathcal{A})$ saying that $\nu_n(a) = 1$ if $n \in \theta(\mathcal{A} \cdot)$ and $\nu_n(a) = 0$ otherwise. Then $\nu_n - \delta_n \rightarrow 0$.

Remark There is a lifting for $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{F}$ iff $\mathcal{A}$ is generated by $\mathcal{F}$ and an algebra $\mathcal{A}_0$ such that every nonempty $\mathcal{A} \in \mathcal{A}_0$ is infinite.
Proposition

Suppose that $\text{fin} \subseteq \mathcal{A} \subseteq P(\omega)$ and the quotient map $\mathcal{A} \to \mathcal{A}/\text{fin}$, $A \to A^\bullet$, admits a lifting. Then $c_0$ is complemented in $C(K_{2\mathfrak{i}})$. 

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**Proof.**

By our assumption there is a homomorphism $\theta : \mathcal{A}/\text{fin} \to \mathcal{A}$, such that $\theta(a)^\bullet = a$ for $a \in \mathcal{A}/\text{fin}$.
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Example: the measure algebra

Let $B = \text{Bor}([0, 1] \setminus \{\lambda = 0\}$ and $S = \text{ult}(B)$ is nonseparable and carries a strictly positive measure.

Frankiewicz & Gutek: Under CH, there is an embedding $\phi: B \to \mathcal{P}(\omega)/\text{fin}$ such that $\lambda(b) = d(\phi(a))$ where $d(\cdot)$ is the usual asymptotic density.

Dow & Hart: Under OCA, $B$ does not embed into $\mathcal{P}(\omega)/\text{fin}$.

Example using $\phi$: $B \to \mathcal{P}(\omega)/\text{fin}$

Let $A = \{A \subseteq \omega; A \cdot \in \phi(B)\}$. Then $K^* A = S = \mathcal{C}(K^* A) = \mathcal{C}(S) \equiv L_\infty([0, 1])$ contains no complemented copy of $c_0$ (is a Grothendieck space).

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$\mathcal{C}(K A)$ contains a complemented copy of $c_0$, spanned by $\chi_{I(n)}$, for some sequence of pairwise disjoint intervals $I(n) \subseteq \omega$. 
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Let $\mathcal{A} = \{A \subseteq \omega; A^* \in \varphi(\mathcal{B})\}$. Then $K^*_{\mathcal{A}} = S$

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Example using $\varphi : \mathcal{B} \rightarrow P(\omega)/\text{fin}$

Let $\mathcal{A} = \{ A \subseteq \omega; A^* \in \varphi(\mathcal{B}) \}$. Then $K_{2\mathfrak{l}}^* = S$

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3. $C(K_{2\mathfrak{l}})$ contains a complemented copy of $c_0$, spanned by $\chi_{I(n)}$, for some sequence of pairwise disjoint intervals $I(n) \subseteq \omega$. 


Our main results

Theorem 1
Assume $p = c$. There is a compactification $\gamma \omega$ such that $\gamma \omega \setminus \omega$ is separable and $c_0$ is not complemented in $C(\gamma \omega)$.

Theorem 2
Assume CH. There is a compactification $\gamma \omega$ such that $\gamma \omega \setminus \omega$ is nonseparable and $c_0$ is complemented in $C(\gamma \omega)$ (so, in particular, $\gamma \omega \setminus \omega$ carries a strictly positive measure).

Question
Does there always exist $\gamma \omega$ such that $\gamma \omega \setminus \omega$ is nonseparable and carries a strictly positive measure?

Remark
There is such $\gamma \omega$ if $b = c$ or $\text{cov}(E) = \omega_1$. 
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Assume CH. There is a compactification $\gamma \omega$ such that $\gamma \omega \setminus \omega$ is nonseparable and $c_0$ is complemented in $C(\gamma \omega)$ (so, in particular, $\gamma \omega \setminus \omega$ carries a strictly positive measure).

Question
Does there always exist $\gamma \omega$ such that $\gamma \omega \setminus \omega$ is nonseparable and carries a strictly positive measure?

Remark
There is such $\gamma \omega$ if $b = c$ or $\text{cov}(E) = \omega_1$. 
**Our main results**

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