Compactifications of ω and the Banach space c_0

Grzegorz Plebanek

Instytut Matematyczny, Universytet Wrocławski

joint work with Piotr Drygier

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Banach spaces

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Notation

- $C(K)$ is the space of continuous functions $K \to \mathbb{R}$.
- c_0 is the space of sequences $x = (x_n)_{n\in\omega}$ converging to 0.
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Classical results

(a) **Sobczyk:** If X is separable then every isomorphic copy of c_0 in X is complemented.

(b) **Phillips:** c_0 is not compemented in l_∞.

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- $\mathscr{X}_c = \mathscr{X}$; examples: compact lines (**Correa & Tausk**).

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$$c_0 = \{g \in C(\gamma\omega) : g|\gamma\omega \setminus \omega \equiv 0\},$$

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Recall that

c_0 is complemented in $C(\gamma\omega)$ whenever $\gamma\omega$ is metrizable.
c_0 is not complemented in $C(\beta\omega)$.

Compactifications of ω and subalgebras of $P(\omega)$

Every zerodimensional $\gamma\omega$ may be seen as the Stone space $\mathrm{ult}(\mathfrak{A})$ of some algebra $\mathfrak{A} \subseteq P(\omega)$ containing *fin*.
We shall write $K_{\mathfrak{A}} = \mathrm{ult}(\mathfrak{A})$ for such a compactification and $K^*_{\mathfrak{A}} = K_{\mathfrak{A}} \setminus \omega$ for its remainder.

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Finitely additive measures

Let $\mathrm{ba}_+(\mathfrak{A})$ denote the space of all bounded finitely additive measures on $\mathfrak{A}$.

$$\mathrm{ba}(\mathfrak{A}) = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathrm{ba}_+(\mathfrak{A})\}$$

is the space of all signed measures. Essentially, $\mathrm{ba}(\mathfrak{A})$ is the dual Banach space of all functionals on $C(K_{\mathfrak{A}})$.

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The following are equivalent for $fin \subseteq \mathfrak{A} \subseteq P(\omega)$

(i) c_0 is complemented in $C(K_{\mathfrak{A}})$;

(ii) there is a uniformly bounded sequence $(\nu_n)_n$ in $\mathrm{ba}(\mathfrak{A})$ such that every ν_n vanishes on *fin* and $\nu_n - \delta_n \to 0$.

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Remarks

- $\nu_n - \delta_n \to 0$ means that for every $A \in \mathfrak{A}$

$$\lim_{n\in A} \nu_n(A) = 1 \text{ and } \lim_{n\in A} \nu_n(\omega \setminus A) = 0.$$

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- If c_0 is complemented in $C(K_{\mathfrak{A}})$ then $K_{\mathfrak{A}}^*$ must carry a strictly positive measure.

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Proposition

Suppose that $fin \subseteq \mathfrak{A} \subseteq P(\omega)$ and the quotient map $\mathfrak{A} \to \mathfrak{A}/fin, A \to A^\bullet$, admits a lifting. Then c_0 is complemented in $C(K_{\mathfrak{A}})$.

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Remark

There is a lifting for $\mathfrak{A} \to \mathfrak{A}/fin$ iff $\mathfrak{A}$ is generated by *fin* and an algebra $\mathfrak{A}_0$ such that every nonempty $A \in \mathfrak{A}_0$ is infinite.

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2. **Drygier & GP:** c_0 is not complemented in $C(K_{\mathfrak{A}})$ though the remainder $K^*_{\mathfrak{A}}$ supports a measure.
3. $C(K_{\mathfrak{A}})$ contains a complemented copy of c_0, spanned by $\chi_{I(n)}$, for some sequence of pairwise disjoint intervals $I(n) \subseteq \omega$.

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Remark

There is such $\gamma\omega$ if $\mathfrak{b} = \mathfrak{c}$ or $\mathrm{cov}(\mathscr{E}) = \omega_1$.