Banach-Mazur games played with arrows

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The setup:

We fix a category $\mathbb{K}$ contained in a bigger category $\mathbb{V}$, such that all sequences in $\mathbb{K}$ have co-limits in $\mathbb{V}$. 

Example

Let $\mathbb{K}$ be the family $T + (X)$ of all nonempty open subsets of a fixed topological space $X$. Let $\mathbb{V}$ be the family of all $G_\delta$ subsets of $X$. 

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Banach-Mazur games

Definition

The Banach-Mazur game on $\mathcal{K}$ is a game played by two players Eve and Odd, with the following rules:

1. Eve starts the game by choosing an object $K_0 \in \mathcal{K}$;
2. Odd responds by choosing an object $K_1 \in \mathcal{K}$ together with a $\mathcal{K}$-arrow $f_1: K_0 \rightarrow K_1$;
3. Eve responds by choosing an object $K_2 \in \mathcal{K}$ and a $\mathcal{K}$-arrow $f_2: K_1 \rightarrow K_2$;
4. and so on...

The result of a play is the co-limit $K_\omega = \lim\{K_n\}_{n \in \omega} \in \text{Obj}(\mathcal{V})$. 

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The result of a play is the co-limit $K_\infty = \lim_{n \in \omega} \{K_n\} \in \text{Obj}(\mathcal{V})$. 

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The Banach-Mazur game on \( \mathcal{K} \) is a game played by two players Eve and Odd, with the following rules:

1. Eve starts the game by choosing an object \( K_0 \in \mathcal{K} \);
2. Odd responds by choosing an object \( K_1 \in \mathcal{K} \) together with a \( \mathcal{K} \)-arrow \( f_0^1 : K_0 \to K_1 \);
3. Eve responds by choosing an object \( K_2 \in \mathcal{K} \) and a \( \mathcal{K} \)-arrow \( f_1^2 : K_1 \to K_2 \);
4. and so on...

\[ K_0 \to K_1 \to K_2 \to \cdots \]
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Definition

The Banach-Mazur game on $\mathcal{K}$ is a game played by two players Eve and Odd, with the following rules:

1. **Eve** starts the game by choosing an object $K_0 \in \mathcal{K}$;
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3. **Eve** responds by choosing an object $K_2 \in \mathcal{K}$ and a $\mathcal{K}$-arrow $f_1^2 : K_1 \to K_2$;
4. and so on...

\[ K_0 \to K_1 \to K_2 \to \cdots \]

The result of a play is the co-limit

\[ K_{\infty} = \lim \{ K_n \}_{n \in \omega} \in \text{Obj}(\mathcal{V}). \]
**Definition**

Let $\mathcal{W}$ be a class of $\mathbf{V}$-objects. We say that **Odd wins** if he has a strategy such that no matter how Eve plays, the co-limit of the resulting sequence is isomorphic to some element of $\mathcal{W}$. Denote this game by $\text{BM}(\mathcal{K}, \mathcal{W})$. 
Theorem

Assume Odd has a winning strategy in $BM(\mathcal{R}, \mathcal{W}_n)$, where each $\mathcal{W}_n$ is closed under isomorphisms. Then Odd has a winning strategy in

$$BM(\mathcal{R}, \bigcap_{n \in \omega} \mathcal{W}_n).$$
Definition

Let $\mathcal{G}$ be another category and let $\Phi: \mathcal{G} \to \mathcal{K}$ be a covariant functor. We say that $\Phi$ is dominating if

(D1) For every $X \in \text{Obj} (\mathcal{K})$ there is $s \in \text{Obj} (\mathcal{G})$ such that $\mathcal{K}(X, \Phi(s)) \neq \emptyset$.

(D2) Given $s \in \text{Obj} (\mathcal{G})$ and $f \in \mathcal{K}$ with $\Phi(s) = \text{dom}(f)$, there exist $g \in \mathcal{G}$ and $h \in \mathcal{K}$ such that $\Phi(g) = h \circ f$.

We say that a subcategory $\mathcal{F}$ of $\mathcal{K}$ is dominating if the inclusion functor $\Phi: \mathcal{F} \to \mathcal{K}$ is dominating.
Let $\mathcal{W} \subseteq \text{Obj}(\mathcal{V})$ and let $\Phi: \mathcal{G} \to \mathcal{K}$ be a dominating functor. Define $\mathcal{U}$ to be the class of all sequences $\vec{s}: \omega \to \mathcal{G}$ satisfying $\lim(\Phi \circ \vec{s}) \in \mathcal{W}$. Then Odd has a winning strategy in $\text{BM}(\mathcal{K}, \mathcal{W})$ if and only if he has a winning strategy in $\text{BM}(\mathcal{G}, \mathcal{U})$. The same applies to Eve.
Let $\mathcal{K} = \mathcal{T}^+(X)$, where $X$ is a compact Hausdorff space. Let $\mathcal{V}$ be the family of all $G_\delta$ subsets of $X$. Define

$$\mathcal{W} = \{\{x\} : x \in X\}.$$
Example I

Let $\mathcal{K} = \mathcal{T}^+(X)$, where $X$ is a compact Hausdorff space. Let $\mathbf{V}$ be the family of all $G_\delta$ subsets of $X$. Define

$$\mathcal{W} = \{\{x\}: x \in X\}.$$ 

**Theorem (Oxtoby)**

*Odd has a winning strategy in $\text{BM}(\mathcal{K}, \mathcal{W})$ if and only if $X$ contains a dense completely metrizable subspace.*
**Definition**

A $V$-object $W$ is **generic** if Odd has a winning strategy in $BM(\mathcal{K}, W)$. 
Definition
A $\mathbf{V}$-object $W$ is generic if Odd has a winning strategy in $\text{BM}(\mathcal{A}, W)$.

Theorem
A generic object, if exists, is unique up to isomorphism.
Theorem

Let $W$ be a generic object and let $X$ be a $V$-object of the form $X = \lim \{X_n\}_{n \in \omega}$ for some sequence $\{X_n\}_{n \in \omega}$ in $K$. Then

$$V(X, W) \neq \emptyset.$$
Fraïssé limits

Definition

A Fraïssé class is a class $\mathcal{F}$ of finitely generated models if a fixed first order language, satisfying the following conditions:

1. For every $A, B \in \mathcal{F}$ there is $D \in \mathcal{F}$ such that both $A$ and $B$ can be embedded into $D$.

2. For every embeddings $f : C \rightarrow A$, $g : C \rightarrow B$ with $C, A, B \in \mathcal{F}$, there exist $E \in \mathcal{F}$ and embeddings $f' : A \rightarrow E$, $g' : B \rightarrow E$ such that

$$f' \circ f = g' \circ g.$$

3. $\mathcal{F}$ has countably many isomorphic types.
Theorem (Fraïssé)

Let $\mathcal{F}$ be a Fraïssé class. Then there exists a unique countably generated model $U$ of the same language as that of $\mathcal{F}$, having the following properties:

1. Every $E \in \mathcal{F}$ embeds into $U$.
2. For every finite set $S \subseteq U$ there is an embedding $e: E \rightarrow U$ such that $E \in \mathcal{F}$ and $S \subseteq e[E]$.
3. For every $E \in \mathcal{F}$, for every two embeddings $f, g: E \rightarrow U$ there exists an automorphism $h: U \rightarrow U$ such that $h \circ f = g$. 

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Theorem

Let $\mathcal{K}$ be a category whose objects form a Fraïssé class $\mathcal{F}$ and arrows are embeddings. Let $U$ be the Fraïssé limit of $\mathcal{F}$. Then Odd has a winning strategy in $BM(\mathcal{K}, U)$. 
Example II

Theorem

Let \( \mathcal{K} \) be the category of all nonempty compact metrizable spaces with continuous surjections and assume that the Banach-Mazur game is played with reversed arrows. Then the \( \mathcal{K} \)-generic compact space is the Cantor set.
Theorem

Let $\mathcal{K}$ be the category of all finite metric spaces with isometric embeddings. Then the $\mathcal{K}$-generic object is the Urysohn universal metric space.
References
