The class of perfectly null sets and its transitive version

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joint work with T. Weiss

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Preliminaries: special subsets in 2^ω

perfectly meager PM

meager in any perfect set (in the subspace topology) [2]

universally null UN

null with respect to any finite Borel diffused measure

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Thm. (Galvin-Mycielski-Solovay). Iff it can be shifted away from any meager set.

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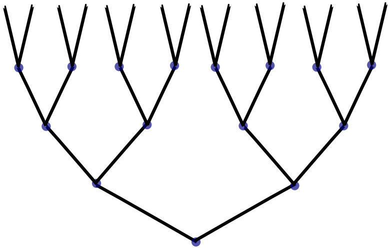
Outline

1. Perfectly null sets
    - definitions
    - simple properties
    - main open problem
2. Perfectly null in the transitive sense sets
    - definitions
    - two theorems
    - open problems

Perfectly null sets: measure on perfect sets



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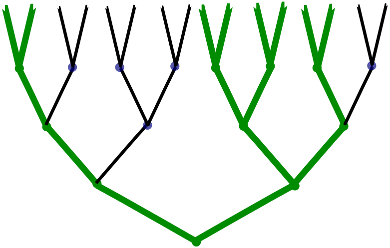
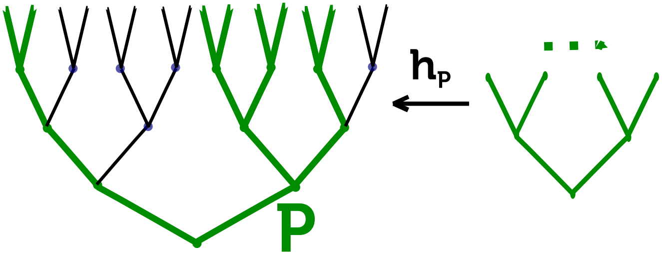


Perfectly null sets: measure on perfect sets

Canonical homeomorphism: $h_P \colon 2^\omega \to P$



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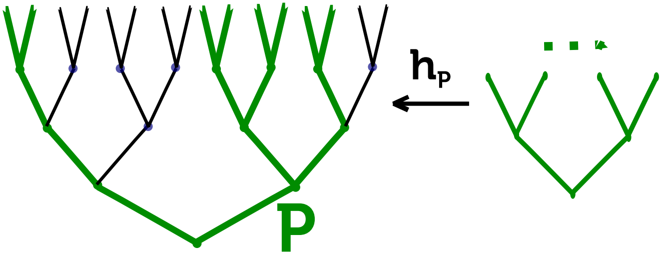

Measure on pefect set $\mu_P(A) = \lambda(h_P^{-1}[A])$,
where λ is the standard Lebesgue measure on 2^ω.

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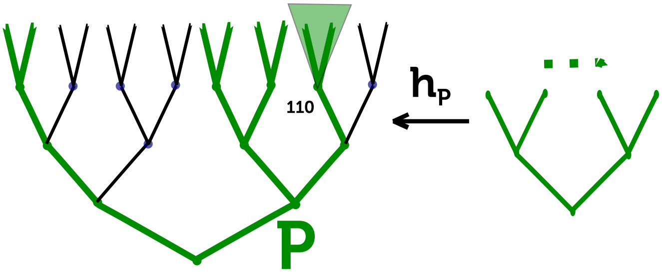

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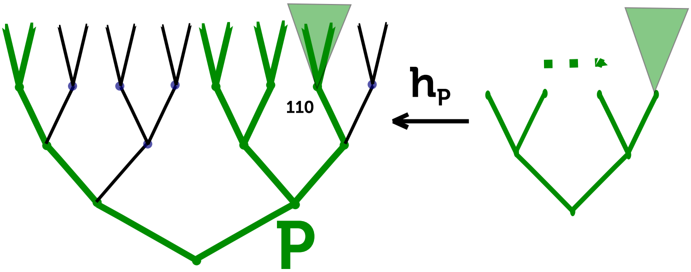

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On the category side all known arguments proving that it is consistent that $\mathrm{UM} \subsetneq \mathrm{PM}$ use the idea of the Lusin function or similar ideas.

Lusin function (Lusin, Sierpiński, [7])

There exists a function $\mathcal{L}\colon \omega^\omega \to 2^\omega$, such that:

- $\mathcal{L}$ is continuous and one-to-one,
- if L is a Lusin set, then $\mathcal{L}[L] \in \mathrm{PM}$,
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Question

Does there exist a measure counterpart to the Lusin function?

The idea of the transitive version

Recall that a set is perfectly meager if it is meager in every perfect set (in the subset topology). It may seem superfluous but we can say that a set X is perfectly meager if for every perfect set P and $t \in 2^\omega$ there exists a F_σ set $F \supseteq X$ such that F is meager in $P + t$. This, and the question of M. Scheepers of whether the algebraic sum of a SN set and a SM set is always s_0, motivates the following definition.

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Theorem (Nowik, Scheepers, Weiss, [4], [5], [3])

SM $\subseteq$ PM′ $\subseteq$ UM and those inclusions are consistently proper.

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Now we would like to know whether

$\mathrm{SN} \subseteq \mathrm{PN}' \subseteq \mathrm{UN}$ and whether those inclusions are consistently proper?

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Proof (sketch):
Let X be a strongly null set and P a perfect set. Recall that since X is strongly null, for every sequence of positive numbers $\langle \varepsilon_n \rangle_{n \in \omega}$ there exists a sequence of open sets $\langle A_n \colon n \in \omega \rangle$ such that
$X \subseteq \bigcap_{m \in \omega} \bigcup_{n \geq m} A_n$ and $\operatorname{diam} A_n \leq \varepsilon_n$.

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We can take such ε_n, that for every A such that $\operatorname{diam} A < \varepsilon_n$, $\mu_P(A) < \frac{1}{2^n}$.
For such ε_n, $(\bigcap_{m\in\omega} \bigcup_{n\geq m} A_n) + t$ is of measure μ_P zero for any $t \in 2^\omega$ and therefore it can be used as the $G_{\delta\sigma}$ set in the definition of perfectly null set in the transitive sense. $\square$

It is consistent that UN $\neq$ PN$'$

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If there exists a UN set of cardinality $\mathfrak{c}$ then there exists a set $Y \in \mathrm{UN} \setminus \mathrm{PN}'$.

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We can construct disjoint perfect sets $C, D \subseteq 2^\omega$, such that $C \cup D$ is linearly independent over $\mathbb{Z}_2$. And we can assume that $X \in UN$, $X \subseteq C$ and $|X| = \mathfrak{c}$.

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Enumerate all $G_{\delta\sigma}$ sets as $\{B_x : x \in X\}$.
Choose $y_x \in x + D$ for $x \in X$, such that $y_x \notin B_x$ if $(x + D) \setminus B_x \neq \varnothing$. Let $Y = \{y_x : x \in X\}$.

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Assume that $Y \in \mathrm{PN}'$. Then there exists $x \in X$ such that $Y \subseteq B_x$ and for any $t \in 2^\omega$ $\mu_D(B_x + t) = 0$. Take $t = x$. We see that $y_x \in B_x$, so $D \cap (B_x + x) = D$, so $\mu_D(B_x + x) = 1$. A contradiction.
$\square$

Open problems

We wanted to know whether $\mathsf{SN} \subseteq \mathsf{PN}' \subseteq \mathsf{UN}$ and whether those inclusions are consistently proper.
We proved that:

1. Every strongly null set is perfectly null in the transitive sense.
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The other two problems are still open:

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In particular, does there exist uncountable PN' set in every model of ZFC?

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References

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