Zero-dimensional spaces as topological and Banach fractals

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X - topological space
$\mathcal{H}(X)$ - the space of nonempty, compact subsets of X

Definition

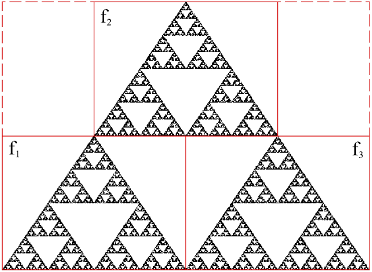
An **Iterated Function System** (IFS) on X is a dynamical system on $\mathcal{H}(X)$ generated by a finite family $\mathcal{F}$ of continuous maps $X \to X$.

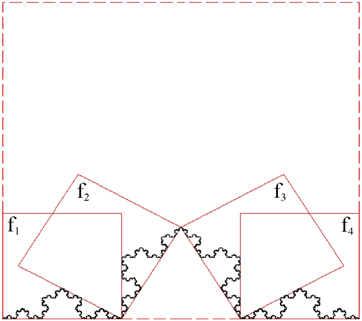
$$K \in \mathcal{H}(X) \qquad \mathcal{F}(K) = \bigcup_{f \in \mathcal{F}} f(K)$$

Definition

The **attractor of the IFS** $\mathcal{F}$ it is a nonempty compact set $A \subset X$ such that $A = \mathcal{F}(A)$ and for every compact set $K \in \mathcal{H}(X)$ the sequence $\left(\mathcal{F}^n(K)\right)_{n=1}^{\infty}$ converges to A in the Vietoris topology on $\mathcal{H}(X)$.

Classical IFS-attractors





Definition

A compact space $X = \bigcup_{f \in \mathcal{F}} f(X)$ for continuous $f \colon X \to X$ is

- **topological fractal** if X is a Hausdorff space and each $f \in \mathcal{F}$ is *topologically contracting*; for every open cover $\mathcal{U}$ of X there is $n \in \mathbb{N}$ such that for any maps $f_1, \dots, f_n \in \mathcal{F}$ the set $f_1 \circ \dots \circ f_n(X) \subset U \in \mathcal{U}$.
- **Banach fractal** if X is metrizable and each $f \in \mathcal{F}$ is a *Banach contraction* with respect to some metric that generates the topology of X.
- **Banach ultrafractal** if X is metrizable, the family $(f(X))_{f \in \mathcal{F}}$ is disjoint and for any $\varepsilon > 0$ each $f \in \mathcal{F}$ has $\mathrm{Lip}(f) < \varepsilon$ with respect to some *ultrametric* generating the topology of X.

A metric d on X is called an *ultrametric* if it satisfies the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for $x, y, z \in X$.

Fact 1

For any compact metrizable space we have the implications

$$\text{Banach ultrafractal} \Rightarrow \text{Banach fractal} \Rightarrow \text{topological fractal}$$

Fact 2

The topology of a compact metrizable space X is generated by an ultrametric if and only if X is zero-dimensional (has a base of closed-and-open sets).

Main theorem

Theorem

For a zero-dimensional compact metrizable space X the following conditions are equivalent:

1. X is a topological fractal;
2. X is a Banach fractal;
3. X is a Banach ultrafractal;
4. the scattered height $\hbar(X)$ of X is not a countable limit ordinal (so, $\hbar(X)$ is either ∞ or a countable successor ordinal).

Scattered height

For a topological space X let

$$X' = \{x \in X : x \text{ is an accumulation point of } X\}$$

be the *Cantor-Bendixson derivative* of X.

- $X^{(\alpha+1)} = (X^{(\alpha)})'$
- $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for a limit ordinal
- $X^{(\infty)} = \bigcap_{\alpha} X^{(\alpha)}$ - the *perfect kernel* of X

Definition

For a scattered topological space X we define its height

$$\hbar(X) = \min\{\beta : X^{(\beta)} \text{ is finite}\}.$$

For an uncountable space X we put $\hbar(X) = \infty$.

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Unital spaces

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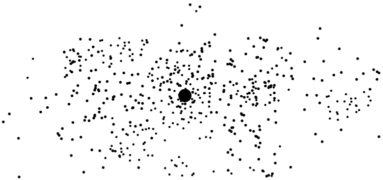
A compact metrizable space X will be called **unital** if X is either uncountable or X is countable and the set $X^{(\hbar(X))}$ is a singleton.



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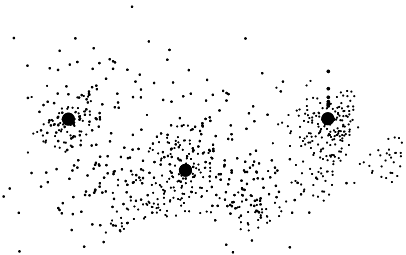

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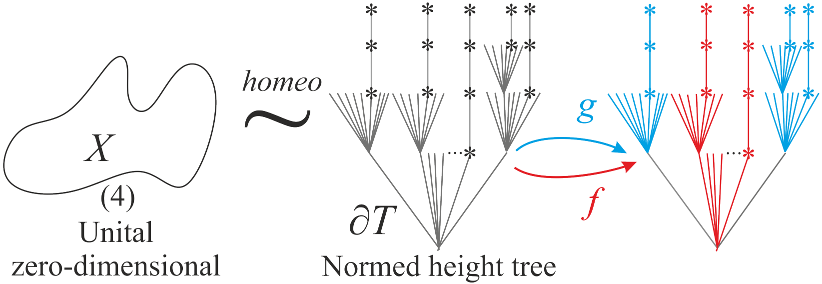
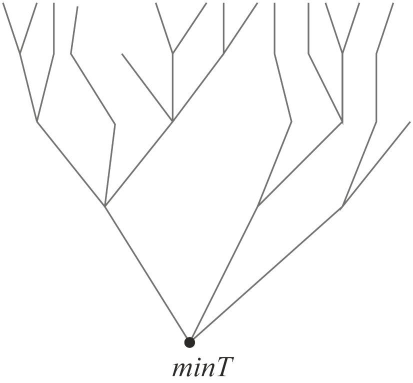
Idea of the proof



Tree

T - **tree** with the (unique) smallest element $\min T$ (*a root*)

- $x \in T$ a set $\mathrm{succ}(x)=$ minimal elements of $\{y \in T : x < y\}$ - the set of **successors** of x
- a **branch** of T - any maximal linearly ordered subset of T
- ∂T the set of all branches of T - **boundary** of the tree T
- for $\bar{x}, \bar{y} \in \partial T$, $\bar{x} \neq \bar{y}$ let $\bar{x} \wedge \bar{y} = \max(\bar{x} \cap \bar{y}) \in T$



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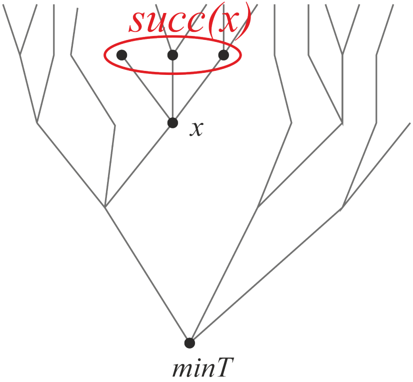

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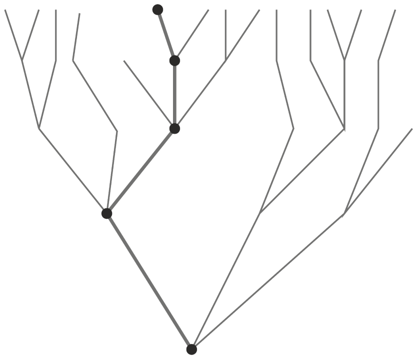

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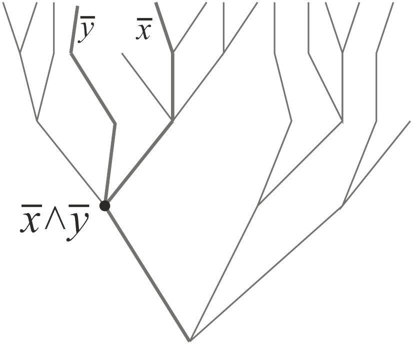

Height tree

$(T, \hbar)$ – the *height tree*

$\hbar : T \to \{-1, \infty\} \cup \omega_1$ – the *height function* on T

such that for every vertex $x \in T$:

- $\mathrm{succ}(x)$ contains exactly one point $*_x$ of height $\hbar(*_x) = -1$;
- if $\hbar(x) \in \{-1, 0\}$, then $\mathrm{succ}(x) = \{*_x\}$ and if $\hbar(x) > 0$, then the set $\mathrm{succ}(x)$ is countable;
- if $\hbar(x) = \infty$, then almost every points of $\mathrm{succ}(x)$ have height ∞;
- if $0 < \hbar(x) < \omega_1$, then

$$\hbar(x) = \sup_{y \in \mathrm{succ}(x)} (\hbar(y) + 1) = \lim_{y \in \mathrm{succ}(x)} (\hbar(y) + 1)$$

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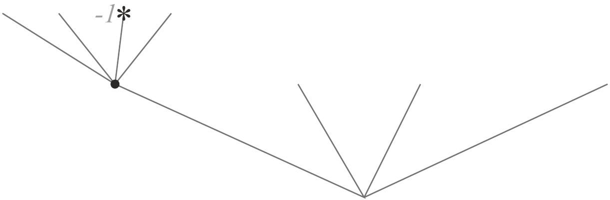

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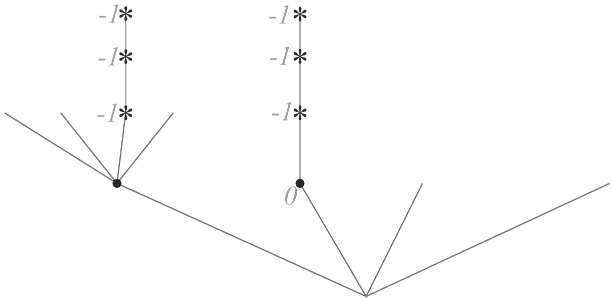

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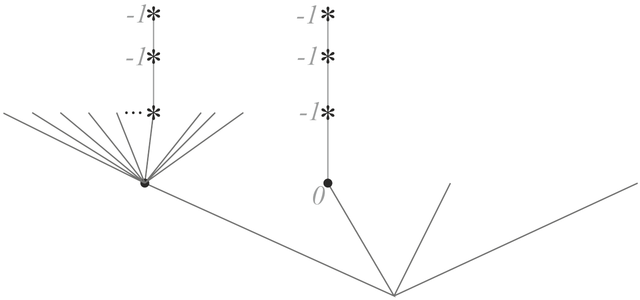

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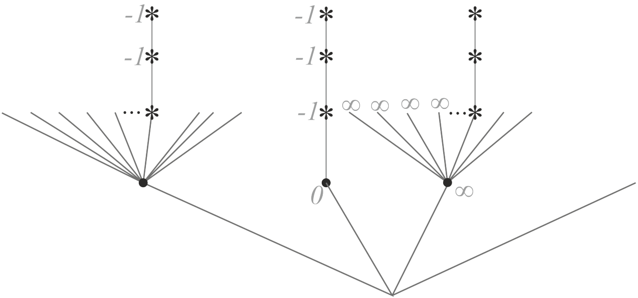
- $\hbar(x) = \infty \quad \Rightarrow$ almost every points of $\operatorname{succ}(x)$ have height ∞



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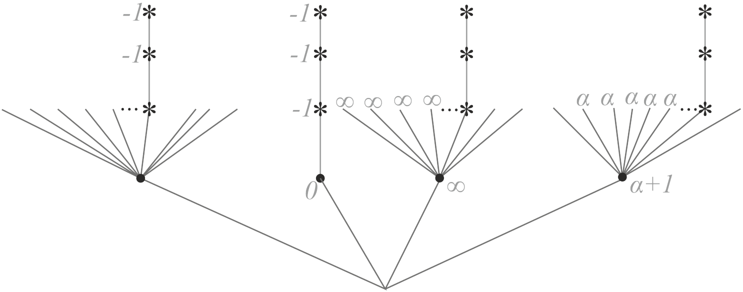

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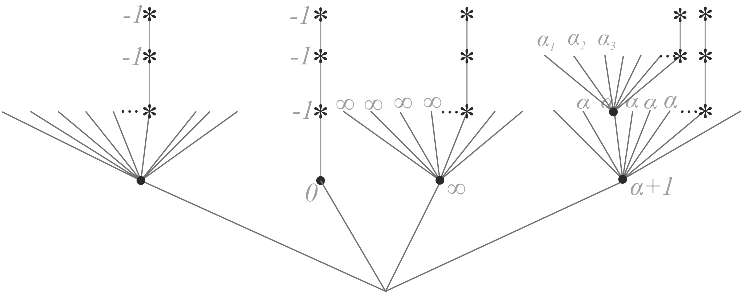


Normed height tree

Definition

A *norm* $\|\cdot\| : T \to \mathbb{R}$ on a height tree T is a function having the following properties:

- for any vertices $x \leq y$ of T we get $\|x\| \geq \|y\| \geq 0$;
- a vertex $x \in T$ has norm $\|x\| = 0$ if and only if $\hbar(x) = -1$;
- $\lim_{x \in T} \|x\| = 0$, which means that for any positive real number ε the set $\{x \in T : \|x\| \geq \varepsilon\}$ is finite.

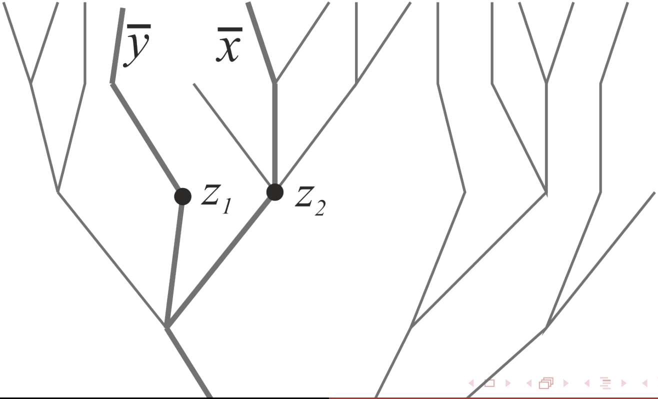
A **normed height tree** is a height tree $(T, \hbar)$ with a norm $\|\cdot\|$.

Canonical ultrametric

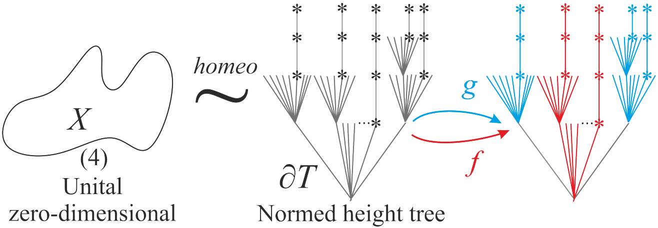
The norm $\|\cdot\|$ of a normed height tree T determines a *canonical ultrametric* d on ∂T defined by

$$d(\bar{x},\bar{y}) = \max\{\|z\| : z \in (\bar{x} \cup \bar{y}) \cap \mathrm{succ}(\bar{x} \wedge \bar{y})\}$$

for any distinct branches $\bar{x}, \bar{y} \in \partial T$.



Step 1 of the proof



Proposition 1

Each unital zero-dimensional compact metrizable space X is homeomorphic to the boundary ∂T of some normed height tree T such that $\hbar(\min T) = \hbar(X)$.

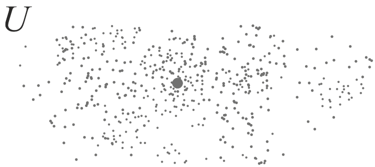
Fix $d \leq 1$
Inductively construct a sequence $(\mathcal{U}_n)_{n\in\omega}$ of covers of X $(\mathcal{U}_{n+1} \succ \mathcal{U}_n)$ containing sets which are

- disjoint
- unital
- diam $\leq 2^{-n+1}$
- points or clopen sets

Construction

Let $\mathcal{U}_0 = \{X\}$

$\mathcal{U}_n \to \mathcal{U}_{n+1}$



Take $U \in \mathcal{U}_n$

$* \in U^{(\hbar(U))}$

Find a disjoint finite cover of U by clopen subsets of diam $\leq 2^{-n}$.

Find a unique set $* \in V$ and choose a neighborhood base $\{V_n\}_{n=1}^{\infty}$ at $*$ consisting of clopen sets such that $V_n \subset V_{n-1}$ for every $n \in \mathbb{N}$.

We can assume that $\lim_{n\to\infty}(\hbar(V_n \setminus V_{n+1}) + 1) = \hbar(V) = \hbar(U)$

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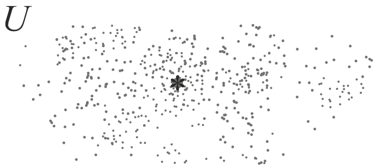

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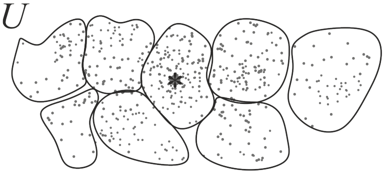

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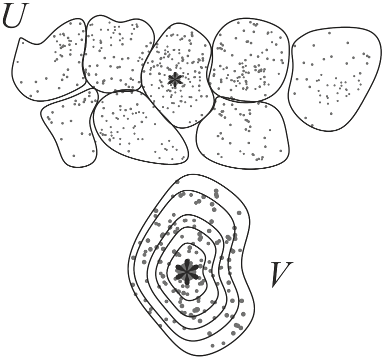

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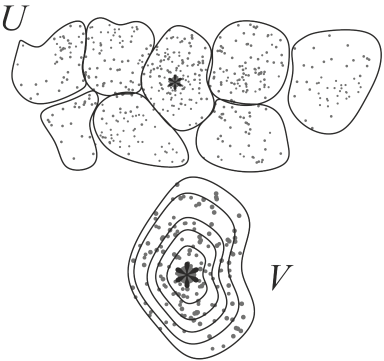

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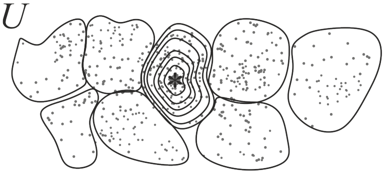

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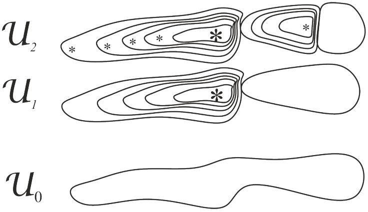
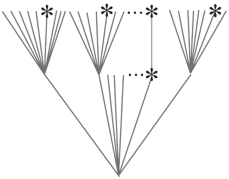
Homeomorphism



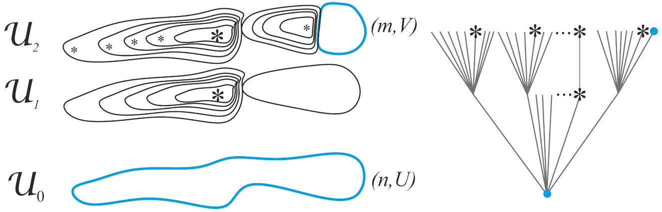


$T = \{(n, U) : n \in \mathbb{N},\ U \in \mathcal{U}_n\}$
where $(n, U) \le (m, V)$ if $n \le m$ and $V \subset U$

$$\hbar(n, U) = \begin{cases} -1, & \text{if } U = \{*\}, \\ \hbar(U), & \text{otherwise.} \end{cases}$$

$\|(n, U)\| = \operatorname{diam}(U \cup \{*\})$

Homeomorphism



$T = \{(n, U) : n \in \mathbb{N},\ U \in \mathcal{U}_n\}$
where $(n, U) \leq (m, V)$ if $n \leq m$ and $V \subset U$

$$\hbar(n, U) = \begin{cases} -1, & \text{if } U = \{*\}, \\ \hbar(U), & \text{otherwise.} \end{cases}$$

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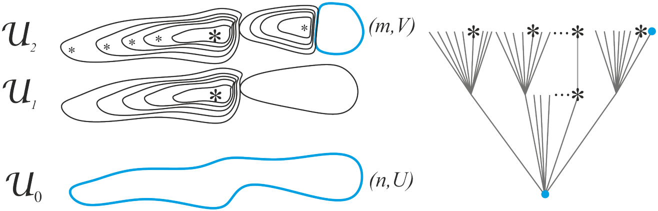

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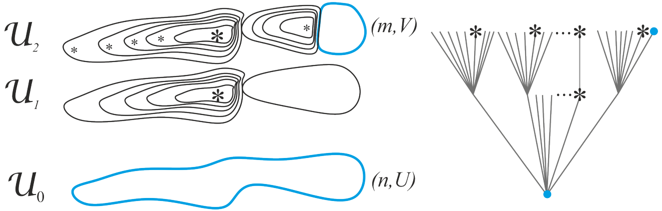

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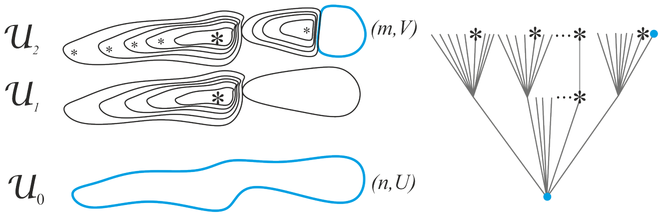

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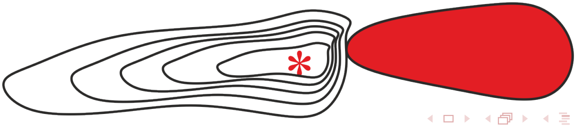
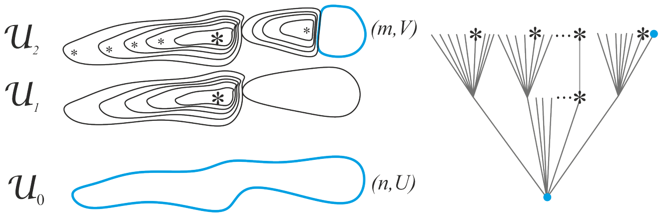


Homeomorphism



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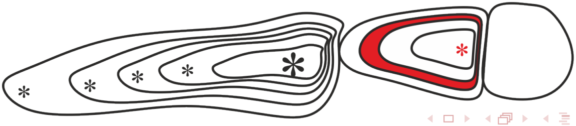
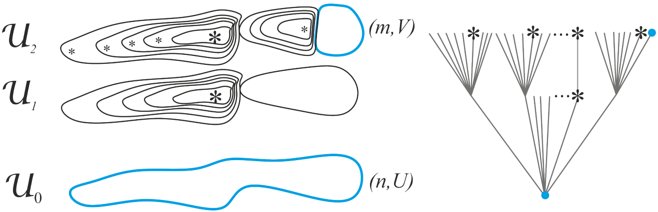


Homeomorphism



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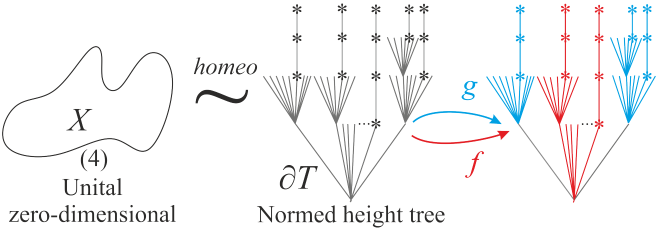
$\|(n, U)\| = \operatorname{diam}(U \cup \{*\})$

$$h : X \to \partial T, \quad h(x) = \{(n, U) : n \in \mathbb{N},\ x \in U\}$$

Step 2 of the proof



Proposition 2

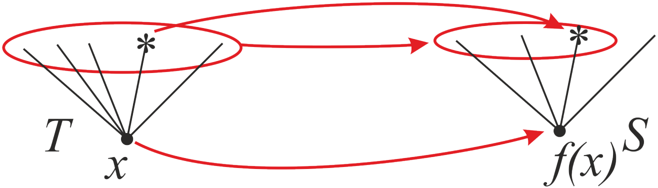
The boundary ∂T of normed height tree T such that $\hbar(\min T)$ is not a limit ordinal, is a Banach ultrafractal.

Height morphism

Definition

For height trees T, S a map $f : T \to S$ is called a **height morphism** if for every $x \in T$ the following conditions are satisfied:

- $\hbar(f(x)) \leq \hbar(x)$,
- $f(\mathrm{succ}(x)) \subset \mathrm{succ}(f(x))$ and $f(*_x) = *_{f(x)}$,
- for each $y \in \mathrm{succ}(f(x)) \setminus \{*_{f(x)}\}$ there is at most one element $z \in \mathrm{succ}(x) \setminus \{*_x\}$ such that $y = f(z)$.



λ-Lipschitz maps

$f\colon T \to S \quad \rightsquigarrow \quad \bar{f} : \partial T \to \partial S$
$\bar{f}(\bar{t}) =$ the unique branch of S containing the linearly ordered set $f(\bar{t}) = \{f(x) : x \in \bar{t}\}$.

Definition

Let T, S be normed height trees. A height morphism $f : T \to S$ is called λ-**Lipschitz** for a real constant λ if $\|f(x)\| \leq \lambda \cdot \|x\|$ for each $x \in T$.

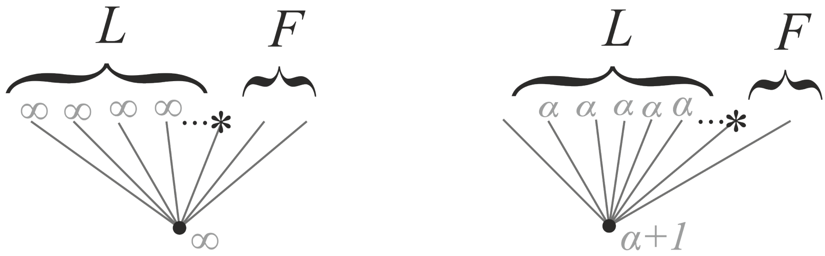
Lemma

Let T, S be normed height trees and λ be a positive real constant. For each λ-Lipschitz height morphism $f : T \to S$, the induced boundary map $\bar{f} : \partial T \to \partial S$ is λ-Lipschitz with respect to the canonical ultrametrics on ∂T and ∂S.

Lemma

For any height trees T, S with $\hbar(\min T) \geq \hbar(\min S)$ there exists a surjective height morphism $f : T \to S$.

∂T is a Banach ultrafractal

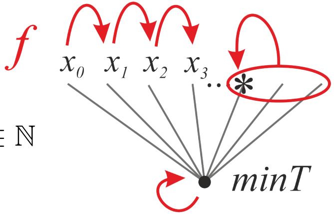


$$L = \{x \in \mathrm{succ}(\min T) : \hbar(x) + 1 = \hbar(\min T)\} = \{x_n\}_{n \in \mathbb{N}}$$

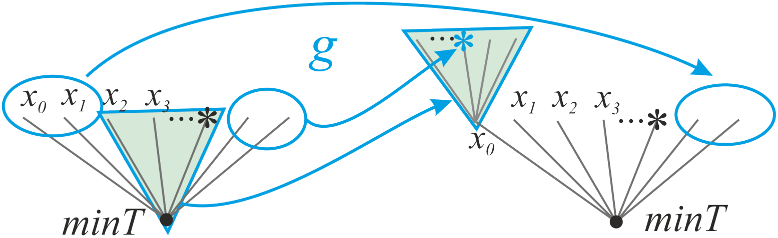
$$F = \mathrm{succ}(\min T) \setminus \left(L \cup \{*_{\min T}\}\right)$$

Function system $\{f, g\}$

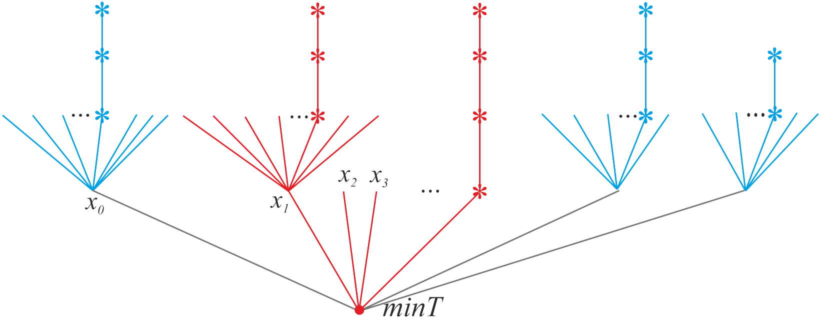
Define a function
$f : \mathrm{succ}(\min T) \to \mathrm{succ}(\min T)$
by the formula



$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ *_{\min T} & \text{otherwise.} \end{cases}$$



$$T = f(T) \cup g(T) \quad \rightsquigarrow \quad \partial T = \bar{f}(\partial T) \cup \bar{g}(\partial T)$$



End of the proof

$$\mathcal{F} = \{f, g\}$$

$$T_{-1} = \emptyset$$
$$T_0 = \{\min T\}$$
$$T_{n+1} = \mathcal{F}^{(n+1)}(\min T) = f(T_n) \cup g(T_n) \text{ for } n \in \mathbb{N}$$

$$\|x\| = \begin{cases} 0 & \text{if } \hbar(x) = -1 \\ \lambda^n & x \in T_n \setminus T_{n-1}. \end{cases}$$

Bibliography

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- M. Nowak, *Topological classification of scattered IFS-attractors*, Topology Appl. **160** (2013), no. 14, 1889–1901.