Zero-dimensional spaces as topological and Banach fractals

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A topological space $X$ is called a topological fractal if $X = \bigcup_{f \in F} f(X)$ for a finite system $F$ of continuous self-maps of $X$ which is topologically contracting in the sense that for every open cover $\mathcal{U}$ of $X$ there is a number $n \in \mathbb{N}$ such that for any functions $f_1, \ldots, f_n \in F$ the set $f_1 \circ \cdots \circ f_n(X)$ is contained in some set $U \in \mathcal{U}$. If, in addition, all functions $f \in F$ has Lipschitz constant $< 1$ with respect to some metric generating the topology of $X$, then the space $X$ is called a Banach fractal. It is known that each topological fractal is compact and metrizable. We prove that a zero-dimensional compact metrizable space $X$ is a topological fractal if and only if $X$ is a Banach fractal if and only if $X$ is either uncountable or $X$ is countable and its scattered height $h(X)$ is a successor ordinal.