On Nikodym's Uniform Boundedness Principle

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If $\mathcal{A}$ is a σ-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

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Theorem (Nikodym's Uniform Boundedness Principle '30)

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A striking improvement of the UBP!

Dunford & Schwartz

The Nikodym Property

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Definition

An infinite Boolean algebra $\mathcal{A}$ has *the Nikodym property* (N) if there are no anti-Nikodym sequences on $\mathcal{A}$.

The Nikodym Property

Notable examples

- σ-algebras (Nikodym '30)

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However, if the Stone space $K_{\mathcal{A}}$ of $\mathcal{A}$ has a convergent sequence, then $\mathcal{A}$ does not have (N):

if $x_n \to x$, then put $\mu_n = n(\delta_{x_n} - \delta_x)$

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$\mathfrak{n} = \min\{|\mathcal{A}| : \text{ infinite } \mathcal{A} \text{ has (N)}\}$.

If $|\mathcal{A}| = \omega$, then $K_{\mathcal{A}} \subseteq 2^{\omega}$, so $\mathcal{A}$ does not have (N). Thus:

$$\omega_1 \leqslant \mathfrak{n} \leqslant \mathfrak{c}.$$

The first bound – the splitting number $\mathfrak{s}$

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$$A \cap B \in [\omega]^\omega \qquad \text{and} \qquad A \setminus B \in [\omega]^\omega.$$

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Theorem (Booth '74)

$\mathfrak{s} = \min\{\kappa :$ *there is a compactum X of weight $w(X) = \kappa$ which is not sequentially compact*$\}$.

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Corollary

$\mathfrak{s} \leqslant \mathfrak{n}$.

The second bound – the bounding number $\mathfrak{b}$

$f \in \omega^\omega$ **dominates** $g \in \omega^\omega$ if $g(n) < f(n)$ for all but finitely many $n \in \omega$.

$\mathcal{F} \subseteq \omega^\omega$ is **dominating** if every $f \in \omega^\omega$ is dominated by some $g \in \mathcal{F}$.

$\mathcal{F}$ is **unbounded** if there is no $f \in \omega^\omega$ dominating every $g \in \mathcal{F}$.

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Proposition

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Barrelled argument

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If $\mathcal{A}$ *has (N), then the space of all simple functions on* $K_{\mathcal{A}}$ *is barrelled.* (Schachermayer '82).

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Constructive argument

By the Josefson–Nissenzweig theorem there exists a sequence $\langle \mu_n : \ n < \omega \rangle$ such that $\|\mu_n\| = 1$ and $\mu_n(a) \to 0$ for every $a \in \mathcal{A}$.

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By the Josefson–Nissenzweig theorem there exists a sequence $\langle \mu_n : n < \omega \rangle$ such that $\|\mu_n\| = 1$ and $\mu_n(a) \to 0$ for every $a \in \mathcal{A}$. If $|\mathcal{A}| < \mathfrak{b}$, then there exists $c \in c_0$ dominating $\langle |\mu_n(a)| : n < \omega \rangle$ for every $a \in \mathcal{A}$.

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If $|\mathcal{A}| < \mathfrak{b}$, then there exists $c \in c_0$ dominating $\langle |\mu_n(a)| : n < \omega \rangle$ for every $a \in \mathcal{A}$.
Put $\nu_n = \mu_n / c_n$.

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Put $\nu_n = \mu_n / c_n$.
$\langle |\nu_n(a)| : n < \omega \rangle$ is bounded for every $a \in \mathcal{A}$ but $\|\nu_n\| \to \infty$.

The lower bounds

Corollary

$\mathfrak{n} \geqslant \max(\mathfrak{b}, \mathfrak{s})$.

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Theorem (Balcar–Pelant–Simon ’80)

It is consistent that $\omega_1 = \mathfrak{s} < \mathfrak{b}$. *(Hence, it is consistent that* $\mathfrak{s} < \mathfrak{n}$*).*

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It is consistent that $\omega_1 = \mathfrak{b} < \mathfrak{s} = \omega_2$. *(Hence, it is consistent that* $\mathfrak{b} < \mathfrak{n}$*).*

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Note that $\mathfrak{d} \geqslant \max(\mathfrak{b}, \mathfrak{s})$. Also note that under Martin's axiom $\mathfrak{b} = \mathfrak{s} = \mathfrak{d} = \mathfrak{c}$, hence $\mathfrak{n} = \mathfrak{d}$ under MA.

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Question

Is it consistent that $\mathfrak{n} < \mathfrak{d}$?

Algebra with (N) and cardinality ω_1

$\mathcal{N}$ – the Lebesgue null ideal

$\mathrm{cof}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{N}$ – cofinal: $\forall A \in \mathcal{N} \exists B \in \mathcal{F} : A \subseteq B\}$

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Assume that $\operatorname{cof}(\mathcal{N}) = \kappa$ for a cardinal number $\kappa < \mathfrak{c}$ such that $\operatorname{cof}([\kappa]^\omega) = \kappa$.

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Assume that $\operatorname{cof}(\mathcal{N}) = \kappa$ for a cardinal number $\kappa < \mathfrak{c}$ such that $\operatorname{cof}([\kappa]^\omega) = \kappa$. Then, there exists a Boolean algebra $\mathcal{B}$ with the Nikodym property and cardinality κ.

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So, if κ as above and $\operatorname{cof}(\mathcal{N}) = \kappa$, then $\mathfrak{n} \leqslant \operatorname{cof}(\mathcal{N})$.

Algebra with (N) and cardinality ω_1

Main Lemma

If $\mathrm{cof}(\mathcal{N}) = \kappa$, then for every countable Boolean algebra $\mathcal{A}$ there exists a family $\{\langle a_n^\gamma \in \mathcal{A} : n \in \omega\rangle : \gamma < \kappa\}$ of κ many antichains in $\mathcal{A}$ with the following property:

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> *for every anti-Nikodym sequence of measures $\langle \mu_n : \ n < \omega\rangle$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n_k : \ k < \omega\rangle$ of naturals such that for every $k < \omega$ the following inequality is satisfied:*

$$|\mu_{n_k} a_k^\gamma| > \sum_{j=0}^{k-1} |\mu_{n_k} a_j^\gamma| + k + 1.$$

Consequence – cofinality of Boolean algebras

Definition

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1. $\omega \leqslant \mathrm{cof}(\mathcal{A}) \leqslant \mathfrak{c}$,
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Theorem (Just–Koszmider '91)

In the Sacks model there exists a Boolean algebra $\mathcal{B}$ *such that* $|\mathcal{B}| = \operatorname{cof}(\mathcal{B}) = \omega_1$.

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Theorem (Pawlikowski–Ciesielski '02)

Assuming $\mathrm{cof}(\mathcal{N}) = \omega_1$, *there exists a Boolean algebra* $\mathcal{B}$ *such that* $|\mathcal{B}| = \mathrm{cof}(\mathcal{B}) = \omega_1$.

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Theorem (Schachermayer '82)

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Corollary

Assuming $\mathrm{cof}(\mathcal{N}) = \kappa$ for κ such that $\mathrm{cof}([\kappa]^\omega) = \kappa$, there exists a Boolean algebra with cardinality κ and cofinality ω_1.

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Assuming $\operatorname{cof}(\mathcal{N}) = \kappa$ for κ such that $\operatorname{cof}([\kappa]^\omega) = \kappa$, there exists a Boolean algebra with cardinality κ and cofinality ω_1.

Question

Is there a consistent example of a Boolean algebra $\mathcal{B}$ for which $\omega_1 < \operatorname{cof}(\mathcal{B}) < \mathfrak{c}$?

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Does there exist a Efimov space?

Fedorčuk: CH, $\diamondsuit$, $\mathfrak{s} = \omega_1$ & $\mathfrak{c} = 2^{\omega_1}$
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Theorem (Pawlikowski–Ciesielski '02, D.S.)

Assuming $\mathrm{cof}(\mathcal{N}) = \omega_1$, there exists a Efimov space.

The end

Thank you for your attention.