On Nikodym’s Uniform Boundedness Principle

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A **measure** \( \mu \) on a Boolean algebra \( \mathcal{A} \) is a signed real-valued finitely additive function of finite variation.

**Theorem (Nikodym’s Uniform Boundedness Principle '30)**

If \( \mathcal{A} \) is a \( \sigma \)-algebra, then every pointwise bounded sequence of measures on \( \mathcal{A} \) is uniformly bounded.
A measure $\mu$ on a Boolean algebra $\mathcal{A}$ is a signed real-valued finitely additive function of finite variation.

A sequence of measures $\langle \mu_n : n < \omega \rangle$ is

- **pointwise bounded** if $\sup_n |\mu_n a| < \infty$ for every $a \in \mathcal{A}$,
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**Theorem (Nikodym’s Uniform Boundedness Principle ’30)**

If $\mathcal{A}$ is a $\sigma$-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.
Nikodym’s Uniform Boundedness Principle

A measure $\mu$ on a Boolean algebra $\mathcal{A}$ is a signed real-valued finitely additive function of finite variation.

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- **pointwise bounded** if $\sup_n |\mu_n a| < \infty$ for every $a \in \mathcal{A}$,
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**Theorem (Nikodym’s Uniform Boundedness Principle ’30)**

*If $\mathcal{A}$ is a $\sigma$-algebra, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.*

A striking improvement of the UBP!

Dunford & Schwartz
A sequence $\langle \mu_n : n < \omega \rangle$ on $A$ is \textit{anti-Nikodym} if it is pointwise bounded on $A$ but not uniformly bounded.

Definition
The Nikodym Property

Definition
A sequence $\langle \mu_n : n < \omega \rangle$ on $\mathcal{A}$ is anti-Nikodym if it is pointwise bounded on $\mathcal{A}$ but not uniformly bounded.

Definition
An infinite Boolean algebra $\mathcal{A}$ has the Nikodym property (N) if there are no anti-Nikodym sequences on $\mathcal{A}$. 
The Nikodym Property

Notable examples

- $\sigma$-algebras (Nikodym '30)

However, if the Stone space $K_A$ of $A$ has a convergent sequence, then $A$ does not have (N): if $x_n \rightarrow x$, then put $\mu_n = n(\delta_{x_n} - \delta_x)$.
The Nikodym Property

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- σ-algebras (Nikodym ’30),
- algebras with Subsequential Completeness Property (Haydon ’81)
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- the algebra of Jordan measurable subsets of $[0, 1]$ (Schachermayer '82; generalized by Wheeler & Graves '83).
The Nikodym Property

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However, if the Stone space $K_\mathcal{A}$ of $\mathcal{A}$ has a convergent sequence, then $\mathcal{A}$ does not have (N):

if $x_n \to x$, then put $\mu_n = n(\delta_{x_n} - \delta_x)$
All *the notable examples* are of cardinality at least $\mathfrak{c}$. 
The Nikodym Number

All *the notable examples* are of cardinality at least $c$.

**Question**

Is there an infinite Boolean algebra with (N) and cardinality less than $c$?
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All the notable examples are of cardinality at least $\mathfrak{c}$.

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Is there an infinite Boolean algebra with (N) and cardinality less than $\mathfrak{c}$?

**The Nikodym number**

$n = \min\{|A| : \text{infinite } A \text{ has (N)}\}$.

If $|A| = \omega$, then $K_A \subseteq 2^\omega$, so $A$ does not have (N). Thus:

$$\omega_1 \leq n \leq \mathfrak{c}.$$
The first bound – the splitting number $s$

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$\mathcal{F} \subseteq [\omega]^\omega$ is **splitting** if for every $A \in [\omega]^\omega$ there exists $B \in \mathcal{F}$ such that:

$$A \cap B \in [\omega]^\omega \quad \text{and} \quad A \setminus B \in [\omega]^\omega.$$  

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Theorem (Booth ’74)

$s = \min\{\kappa : \text{there is a compactum } X \text{ of weight } w(X) = \kappa \text{ which is not sequentially compact}\}$. 
The first bound – the splitting number $\mathcal{S}$

If the Stone space $K_{\mathcal{A}}$ of $\mathcal{A}$ has a convergent sequence, then $\mathcal{A}$ does not have (N).

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**Corollary**

$$\mathcal{S} \leq \aleph_0.$$
The second bound – the bounding number $b$ 

$f \in \omega^\omega$ dominates $g \in \omega^\omega$ if $g(n) < f(n)$ for all but finitely many $n \in \omega$.

$\mathcal{F} \subseteq \omega^\omega$ is dominating if every $f \in \omega^\omega$ is dominated by some $g \in \mathcal{F}$.

$\mathcal{F}$ is unbounded if there is no $f \in \omega^\omega$ dominating every $g \in \mathcal{F}$. 

\[ b = \min\{ |\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is dominating} \} \]

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Proposition $b \leq n$. 

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On Nikodym’s Uniform Boundedness Principle
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**Proposition**

$b \leq n$. 

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On Nikodym's Uniform Boundedness Principle
Barrelled argument

All metrizable barrelled spaces have dimension at least $\mathfrak{b}$. (Saxon–Sanchez-Ruiz ’96)
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All metrizable barrelled spaces have dimension at least $b$. (Saxon–Sanchez-Ruiz ’96)

If $A$ has $(N)$, then the space of all simple functions on $K_A$ is barrelled. (Schachermayer ’82).
The second bound – the bound number $b$

**Barrelled argument**

*All metrizable barrelled spaces have dimension at least $b$.*  
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*If $\mathcal{A}$ has $(N)$, then the space of all simple functions on $K_{\mathcal{A}}$ is barrelled.*  
(Schachermayer ’82).

**Constructive argument**

By the Josefson–Nissenzweig theorem there exists a sequence $\langle \mu_n : n < \omega \rangle$ such that $\|\mu_n\| = 1$ and $\mu_n(a) \to 0$ for every $a \in \mathcal{A}$.
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### Constructive argument

By the Josefson–Nissenzweig theorem there exists a sequence $\langle \mu_n : n < \omega \rangle$ such that $\|\mu_n\| = 1$ and $\mu_n(a) \to 0$ for every $a \in \mathcal{A}$. If $|\mathcal{A}| < b$, then there exists $c \in c_0$ dominating $\langle |\mu_n(a)| : n < \omega \rangle$ for every $a \in \mathcal{A}$. 

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**The second bound – the bound number $b$**

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**Damian Sobota**

**On Nikodym’s Uniform Boundedness Principle**
The second bound – the bound number $\mathfrak{b}$

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By the Josefson–Nissenzweig theorem there exists a sequence $\langle \mu_n : n < \omega \rangle$ such that $\|\mu_n\| = 1$ and $\mu_n(a) \to 0$ for every $a \in \mathcal{A}$. If $|\mathcal{A}| < \mathfrak{b}$, then there exists $c \in c_0$ dominating $\langle |\mu_n(a)| : n < \omega \rangle$ for every $a \in \mathcal{A}$. Put $\nu_n = \mu_n / c_n$. 

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Put $\nu_n = \mu_n / c_n$.

$\langle |\nu_n(a)| : n < \omega \rangle$ is bounded for every $a \in \mathcal{A}$ but $\|\nu_n\| \to \infty$. 
The lower bounds

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The lower bounds

**Corollary**

\[ n \geq \max(b, s). \]

**Theorem (Balcar–Pelant–Simon ’80)**

*It is consistent that* \( \omega_1 = s < b \). (Hence, it is consistent that \( s < n \)).

Note that \( d = \max(b, s) \).

Also note that under Martin’s axiom \( b = s = d = c \), hence \( n = d \) under MA.

**Question**

Is it consistent that \( n < d \)?
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Note that $\mathfrak{d} \geq \max(b, s)$. Also note that under Martin’s axiom $b = s = \mathfrak{d} = \mathfrak{c}$, hence $n = \mathfrak{d}$ under MA.
The lower bounds

Corollary

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Note that \( d \geq \max(b, s) \). Also note that under Martin’s axiom \( b = s = d = c \), hence \( n = d \) under MA.

Question

Is it consistent that \( n < d \)?
$\mathcal{N}$ – the Lebesgue null ideal

$$\text{cof}(\mathcal{N}) = \min\{|F| : F \subseteq \mathcal{N} \text{ – cofinal: } \forall A \in \mathcal{N} \exists B \in F : A \subseteq B\}$$
Algebra with (N) and cardinality $\omega_1$

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Note that $\omega_1 \leq \varnothing \leq \text{cof}(\mathcal{N}) \leq c$. 
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**Theorem (D.S.)**

Assume that $\operatorname{cof}(\mathcal{N}) = \kappa$ for a cardinal number $\kappa < \mathfrak{c}$ such that $\operatorname{cof}([\kappa]^\omega) = \kappa$. 
\( \mathcal{N} \) – the Lebesgue null ideal

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Note that $\omega_1 \leq \mathfrak{d} \leq \text{cof}(\mathcal{N}) \leq c$.

**Theorem (D.S.)**

Assume that $\text{cof}(\mathcal{N}) = \kappa$ for a cardinal number $\kappa < c$ such that $\text{cof}([\kappa]^{\omega}) = \kappa$. Then, there exists a Boolean algebra $\mathcal{B}$ with the Nikodym property and cardinality $\kappa$.

So, if $\kappa$ as above and $\text{cof}(\mathcal{N}) = \kappa$, then $\mathfrak{n} \leq \text{cof}(\mathcal{N})$. 
Main Lemma

If $\text{cof}(\mathcal{N}) = \kappa$, then for every countable Boolean algebra $\mathcal{A}$ there exists a family $\{\langle a_\gamma^n \in \mathcal{A} : n \in \omega \rangle : \gamma < \kappa\}$ of $\kappa$ many antichains in $\mathcal{A}$ with the following property:

For every anti-Nikodym sequence of measures $\langle \mu_n : n < \omega \rangle$ there exist $\gamma < \kappa$ and an increasing sequence $\langle n_k : k < \omega \rangle$ of naturals such that for every $k < \omega$ the following inequality is satisfied:

$$|\mu_{n_k} a_\gamma^n| > k - 1 \sum_{j=0}^{k} |\mu_{n_k} a_\gamma^j| + k + 1.$$
Main Lemma

If \( \text{cof}(\mathcal{N}) = \kappa \), then for every countable Boolean algebra \( \mathcal{A} \) there exists a family \( \{ \langle a_\gamma^n \in \mathcal{A} : n \in \omega \rangle : \gamma < \kappa \} \) of \( \kappa \) many antichains in \( \mathcal{A} \) with the following property:

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\[
|\mu_{n_k} a_k^\gamma| > \sum_{j=0}^{k-1} |\mu_{n_k} a_j^\gamma| + k + 1.
\]
Consequence – cofinality of Boolean algebras

**Definition**

\[ \text{cof}(\mathcal{A}) = \min\{\kappa : \exists \langle \mathcal{A}_\xi : \xi < \kappa \rangle \uparrow \mathcal{A}\} . \]

**Theorem (Koppelberg '77)**

\[ \omega / \leq \text{cof}(\mathcal{A}) / \leq c \]

**Theorem (Just–Koszmider '91)**

In the Sacks model there exists a Boolean algebra \( \mathcal{B} \) such that \( |\mathcal{B}| = \text{cof}(\mathcal{B}) = \omega_1 \).

**Theorem (Pawlikowski–Ciesielski '02)**

Assuming \( \text{cof}(\mathcal{N}) = \omega_1 \), there exists a Boolean algebra \( \mathcal{B} \) such that \( |\mathcal{B}| = \text{cof}(\mathcal{B}) = \omega_1 \).
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**Theorem (Koppelberg ’77)**

1. \( \omega \leq \text{cof}(\mathcal{A}) \leq \mathfrak{c}, \)
2. (MA) If \( |\mathcal{A}| < \mathfrak{c}, \) then \( \text{cof}(\mathcal{A}) = \omega. \)
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cof(\mathcal{A}) = \min\{\kappa : \exists \langle A_\xi : \xi < \kappa \rangle \uparrow \mathcal{A}\}.

Theorem (Koppelberg ’77)
1. \omega \leq \text{cof}(\mathcal{A}) \leq c,
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In the Sacks model there exists a Boolean algebra \mathcal{B} such that |\mathcal{B}| = \text{cof}(\mathcal{B}) = \omega_1.
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1. \( \omega \leq \text{cof}(\mathcal{A}) \leq c \),
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**Theorem (Pawlikowski–Ciesielski ’02)**

Assuming \( \text{cof}(\mathcal{N}) = \omega_1 \), there exists a Boolean algebra \( \mathcal{B} \) such that \( |\mathcal{B}| = \text{cof}(\mathcal{B}) = \omega_1 \).
Theorem (Schachermayer ’82)

If $\mathcal{A}$ has the Nikodym property, then $\text{cof}(\mathcal{A}) > \omega$. 

Corollary

Assuming $\text{cof}(\mathcal{N}) = \kappa$ for $\kappa$ such that $\text{cof}(\kappa \omega) = \kappa$, there exists a Boolean algebra with cardinality $\kappa$ and cofinality $\omega_1$.

Question

Is there a consistent example of a Boolean algebra $\mathcal{B}$ for which $\omega_1 < \text{cof}(\mathcal{B}) < \text{c}$?
Theorem (Schachermayer ’82)

If $A$ has the Nikodym property, then $\text{cof}(A) > \omega$.

Corollary

Assuming $\text{cof}(\mathcal{N}) = \kappa$ for $\kappa$ such that $\text{cof}([\kappa]^\omega) = \kappa$, there exists a Boolean algebra with cardinality $\kappa$ and cofinality $\omega_1$. 
Theorem (Schachermayer ’82)

If $A$ has the Nikodym property, then $\text{cof}(A) > \omega$.

Corollary

Assuming $\text{cof}(\mathcal{N}) = \kappa$ for $\kappa$ such that $\text{cof}([\kappa]^{\omega}) = \kappa$, there exists a Boolean algebra with cardinality $\kappa$ and cofinality $\omega_1$.

Question

Is there a consistent example of a Boolean algebra $B$ for which $\omega_1 < \text{cof}(B) < c$?
Definition

An infinite compact Hausdorff space is a **Efimov space** if it contains neither a convergent sequence nor a copy of $\beta \omega$. 
Consequence – the Efimov problem

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The Efimov Problem ’69
Does there exist a Efimov space?
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**The Efimov Problem ’69**

Does there exist a Efimov space?

Fedorčuk: CH, $\diamondsuit$, $s = \omega_1$ & $c = 2^{\omega_1}$

Dow: $\text{cof}([s]^{\omega}) = s$ & $2^s < 2^c$

and many more...
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Theorem (Pawlikowski–Ciesielski ’02, D.S.)

**Assuming** $\text{cof}(\mathcal{N}) = \omega_1$, **there exists a Efimov space.**
Thank you for your attention.