Borel chromatic numbers

José de Jesús Pelayo Gómez

Posgrado Conjunto en Ciencias Matemáticas
UNAM-UMSNH
Morelia, México

Winter School in Abstract Analysis
Hejnice, Czech Republic.

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation.

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation. An element of V is called a vertex and elements of E are edges. We say that vertex v is adyacent to u iff $\{u, v\} \in E$.

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation. An element of V is called a vertex and elements of E are edges. We say that vertex v is adyacent to u iff $\{u, v\} \in E$.

- We could think a graph as a set of vertices and an irreflexive and symmetric relation.

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation. An element of V is called a vertex and elements of E are edges. We say that vertex v is adyacent to u iff $\{u, v\} \in E$.
- We could think a graph as a set of vertices and an irreflexive and symmetric relation.
- Graphs that we consider are graphs on *Polish spaces* (V is a Polish space).

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation. An element of V is called a vertex and elements of E are edges. We say that vertex v is adyacent to u iff $\{u, v\} \in E$.
- We could think a graph as a set of vertices and an irreflexive and symmetric relation.
- Graphs that we consider are graphs on *Polish spaces* (V is a Polish space).
- A coloring is a function $c : V \to k$ s.t. $c(u) \neq c(v)$ if u is adyacent to v. In that case we say that c is a coloring with k colors.

- A graph G is a pair (V, E), where V is a non empty set and $E \subseteq [V]^2$. In this talk I always use this notation. An element of V is called a vertex and elements of E are edges. We say that vertex v is adyacent to u iff $\{u, v\} \in E$.

- We could think a graph as a set of vertices and an irreflexive and symmetric relation.

- Graphs that we consider are graphs on *Polish spaces* (V is a Polish space).

- A coloring is a function $c : V \to k$ s.t. $c(u) \neq c(v)$ if u is adyacent to v. In that case we say that c is a coloring with k colors.

- Note that always there exists colorings for every graph because we could put $k = |V|$.

- The chromatic number of a graph G is the minimum k such that there exists a coloring with k colors and we detone it by $\chi(G)$.

- The chromatic number of a graph G is the minimum k such that there exists a coloring with k colors and we detone it by $\chi(G)$.

- The Borel chromatic number of a graph G is the minimun k for that exists a Borel coloring with k colors. We are thinking k with the discrete topology and denote the Borel chromatic number of G by $\chi_B(G)$.

- The chromatic number of a graph G is the minimum k such that there exists a coloring with k colors and we detone it by $\chi(G)$.

- The Borel chromatic number of a graph G is the minimun k for that exists a Borel coloring with k colors. We are thinking k with the discrete topology and denote the Borel chromatic number of G by $\chi_B(G)$.

- The idea of this talk is present Borel versions of theorems regarding the chromatic number. It was extracted from an article due to Kechris, Solecki and Todorcevic.

Proposition (finite version)

Let G be a finite graph. If every vertex in G has at most n adyacent vertices then $\chi(G) \leq n+1$.

Proposition (finite version)

Let G be a finite graph. If every vertex in G has at most n adyacent vertices then $\chi(G) \leq n+1$.

Proof

- By induction over n. The case $n = 0$ is obvious because we have a graph without edges.

Proposition (finite version)

Let G be a finite graph. If every vertex in G has at most n adyacent vertices then $\chi(G) \leq n+1$.

Proof

- By induction over n. The case $n = 0$ is obvious because we have a graph without edges.
- For $n > 0$ we enumerate $V = \{v_i : i \in |V|\}$ and define A_i for $i \in |V|$ as follows:
  - $A_0 = \{v_0\}$.
  - $A_{i+1} = A_i \cup \{v_{i+1}\}$ if v_{i+1} is not adyacent to any vertex in A_i and $A_{i+1} = A_i$ in other case.

Proposition (finite version)

Let G be a finite graph. If every vertex in G has at most n adyacent vertices then $\chi(G) \leq n+1$.

Proof

- By induction over n. The case $n=0$ is obvious because we have a graph without edges.
- For $n>0$ we enumerate $V=\{v_i : i \in |V|\}$ and define A_i for $i \in |V|$ as follows:
  - $A_0 = \{v_0\}$.
  - $A_{i+1} = A_i \cup \{v_{i+1}\}$ if v_{i+1} is not adyacent to any vertex in A_i and $A_{i+1} = A_i$ in other case.
- Every vertex in G is in $A_{|V|-1}$ or has an edge with some vertex in $A_{|V|-1}$. Then we could color $A_{|V|-1}$ with one color and apply induction hypothesis to $G - A_{|V|-1}$. ■

Proposition (finite version)

Let G be a finite graph. If every vertex in G has at most n adyacent vertices then $\chi(G) \leq n+1$.

Proof

- By induction over n. The case $n = 0$ is obvious because we have a graph without edges.
- For $n > 0$ we enumerate $V = \{v_i : i \in |V|\}$ and define A_i for $i \in |V|$ as follows:
  - $A_0 = \{v_0\}$.
  - $A_{i+1} = A_i \cup \{v_{i+1}\}$ if v_{i+1} is not adyacent to any vertex in A_i and $A_{i+1} = A_i$ in other case.
- Every vertex in G is in $A_{|V|-1}$ or has an edge with some vertex in $A_{|V|-1}$. Then we could color $A_{|V|-1}$ with one color and apply induction hypothesis to $G - A_{|V|-1}$. ■
- Remark: in ZFC $\chi(G) \leq n$ iff $\chi(F) \leq n$ for every finite subgraph (De Bruijn–Erdős theorem). So we have an infinite version of the theorem.

Proposition (Borel version) (Kechis,Solecki and Todorcevic)

Let G be a graph. If every vertex in G has at most n adyacent vertices and $E[Y] = \{v \in V : (\exists y \in Y)(\{v, y\} \in E)\}$ is Borel for every Borel set $Y \subseteq V$, then $\chi_B(G) \leq n + 1$.

Proposition (Borel version) (Kechis,Solecki and Todorcevic)

Let G be a graph. If every vertex in G has at most n adyacent vertices and $E[Y] = \{v \in V : (\exists y \in Y)(\{v, y\} \in E)\}$ is Borel for every Borel set $Y \subseteq V$, then $\chi_B(G) \leq n + 1$.

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G using that the topology is second countable.

Proposition (Borel version) (Kechis,Solecki and Todorcevic)

Let G be a graph. If every vertex in G has at most n adyacent vertices and $E[Y] = \{v \in V : (\exists y \in Y)(\{v, y\} \in E)\}$ is Borel for every Borel set $Y \subseteq V$, then $\chi_B(G) \leq n + 1$.

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G using that the topology is second countable.
- The proof is the same as in the finite version but we need to construct a Borel independent set in another way. Take a partition of independent sets $\{B_n : n \in \omega\}$ which is possible because $\chi_B(G) \leq \omega$. Define:
  - $A_0 = B_0$.
  - $A_{n+1} = A_n \cup (B_{n+1} \setminus E[A_n])$.
  - $A = \bigcup_n A_n$

Proposition (Borel version) (Kechis,Solecki and Todorcevic)

Let G be a graph. If every vertex in G has at most n adyacent vertices and $E[Y] = \{v \in V : (\exists y \in Y)(\{v, y\} \in E)\}$ is Borel for every Borel set $Y \subseteq V$, then $\chi_B(G) \leq n + 1$.

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G using that the topology is second countable.
- The proof is the same as in the finite version but we need to construct a Borel independent set in another way. Take a partition of independent sets $\{B_n : n \in \omega\}$ which is possible because $\chi_B(G) \leq \omega$. Define:
  - $A_0 = B_0$.
  - $A_{n+1} = A_n \cup (B_{n+1} \setminus E[A_n])$.
  - $A = \bigcup_n A_n$
- Using induction and extending the topology s.t. A becomes clopen we finish the proof. ∎

Some remarks

- The previous version of the theorem is the best posible, for example in $\mathcal{K}_{n+1}$ the bigger degree is n and $\chi(\mathcal{K}_{n+1}) = n + 1$ but there are graphs in which every vertex has infinite degree and $\chi(B)$ is finite.

Some remarks

- The previous version of the theorem is the best posible, for example in $\mathcal{K}_{n+1}$ the bigger degree is n and $\chi(\mathcal{K}_{n+1}) = n+1$ but there are graphs in which every vertex has infinite degree and $\chi(B)$ is finite.

- Given a function $f : V \to V$, $G_f = (V, E)$ where uEv iff $u \neq v$ and $f(u) = v$ or $f(v) = u$.

Proposition

Let $f : V \to V$ be a function. Then $\chi(G_f) \leq 3$

Proposition

Let $f : V \to V$ be a function. Then $\chi(G_f) \leq 3$

Proof

- When $|V| \in \omega$ we do the proof by induction. For $|V| \leq 3$ is obvious.

Proposition

Let $f : V \to V$ be a function. Then $\chi(G_f) \leq 3$

Proof

- When $|V| \in \omega$ we do the proof by induction. For $|V| \leq 3$ is obvious.
- Suppose $|V| > 3$. There is a vertex u of degree 0, 1 or 2 by pigeonhole principle.

Proposition

Let $f : V \to V$ be a function. Then $\chi(G_f) \leq 3$

Proof

- When $|V| \in \omega$ we do the proof by induction. For $|V| \leq 3$ is obvious.
- Suppose $|V| > 3$. There is a vertex u of degree 0, 1 or 2 by pigeonhole principle.
- If we remove u from G we use hypothesis induction (maybe some $f(v) = u$ and in that case we modify $f|V \setminus u$), then we can paint $G - v$ with three colors. It's easy to color V with three colors.

Proposition

Let $f : V \to V$ be a function. Then $\chi(G_f) \leq 3$

Proof

- When $|V| \in \omega$ we do the proof by induction. For $|V| \leq 3$ is obvious.
- Suppose $|V| > 3$. There is a vertex u of degree 0, 1 or 2 by pigeonhole principle.
- If we remove u from G we use hypothesis induction (maybe some $f(v) = u$ and in that case we modify $f|V \setminus u$), then we can paint $G - v$ with three colors. It's easy to color V with three colors.
- If V is infinite we use again the compactness theorem.■

Proposition (Borel version) (Kechris, Solecki and Todorcevic) (proof by Palamourdas)

Let $f : V \to V$ be a Borel function. Then $\chi_B(G_f) \leq 3$ or $\chi_B(G_f) = \omega$

Proposition (Borel version) (Kechris, Solecki and Todorcevic) (proof by Palamourdas)

Let $f : V \to V$ be a Borel function. Then $\chi_B(G_f) \leq 3$ or $\chi_B(G_f) = \omega$

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G_f.

Proposition (Borel version) (Kechris, Solecki and Todorcevic) (proof by Palamourdas)

Let $f : V \to V$ be a Borel function. Then $\chi_B(G_f) \leq 3$ or $\chi_B(G_f) = \omega$

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G_f.
- Suppose that $\chi_B(G_f) < \omega$ and take a Borel partition $A_0, A_1, ..., A_{n-1}$ for some n.

Proposition (Borel version) (Kechris, Solecki and Todorcevic) (proof by Palamourdas)

Let $f : V \to V$ be a Borel function. Then $\chi_B(G_f) \leq 3$ or $\chi_B(G_f) = \omega$

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G_f.
- Suppose that $\chi_B(G_f) < \omega$ and take a Borel partition $A_0, A_1, ..., A_{n-1}$ for some n.
- Define recursively B_i and C_i as follows:
  - $B_0 = A_0$ and $C_0 = \emptyset$,
  - $B_{i+1} = B_i \cup \{x \in A_{i+1} : f(x) \notin B_i\}$ and
  - $C_{i+1} = C_i \cup \{x \in A_{i+1} : f(x) \in B_i\}$

Proposition (Borel version) (Kechris, Solecki and Todorcevic) (proof by Palamourdas)

Let $f : V \to V$ be a Borel function. Then $\chi_B(G_f) \leq 3$ or $\chi_B(G_f) = \omega$

Proof

- It's not hard to see that there exists a Borel coloring with ω colors for G_f.
- Suppose that $\chi_B(G_f) < \omega$ and take a Borel partition $A_0, A_1, ..., A_{n-1}$ for some n.
- Define recursively B_i and C_i as follows:
  - $B_0 = A_0$ and $C_0 = \emptyset$,
  - $B_{i+1} = B_i \cup \{x \in A_{i+1} : f(x) \notin B_i\}$ and
  - $C_{i+1} = C_i \cup \{x \in A_{i+1} : f(x) \in B_i\}$
- Let $B = B_{n-1}$ and $C = C_{n-1}$. Note that $V = B \cup C$ and B and C are Borel. Our claim is that C is independent and B could be colored by two colors.

Proof of claim

- Suppose $f(u) = v$ and $u \neq v$. $u \in A_i$ and $v \in A_j$ and $i \neq j$ because $\{A_i : i \in n\}$ is a coloring.

Proof of claim

- Suppose $f(u) = v$ and $u \neq v$. $u \in A_i$ and $v \in A_j$ and $i \neq j$ because $\{A_i : i \in n\}$ is a coloring.
- If $u \in C$ then $f(u) \in B_{i-1}$ and so $v \notin C$. We conclude that C is an independent set.

Proof of claim

- Suppose $f(u) = v$ and $u \neq v$. $u \in A_i$ and $v \in A_j$ and $i \neq j$ because $\{A_i : i \in n\}$ is a coloring.
- If $u \in C$ then $f(u) \in B_{i-1}$ and so $v \notin C$. We conclude that C is an independent set.
- For $u \in B$ there are three cases:
  - For every k $f^k(u) \in B$ and $f^{n-1}(u) = f^n(u)$ for some n.
  - For every k $f^k(u) \in B$ and $f^{n-1}(u) \neq f^n(u)$ for every n.
  - There exists n s.t. $f^n(u) \in C$.

Proof of claim

- Suppose $f(u) = v$ and $u \neq v$. $u \in A_i$ and $v \in A_j$ and $i \neq j$ because $\{A_i : i \in n\}$ is a coloring.
- If $u \in C$ then $f(u) \in B_{i-1}$ and so $v \notin C$. We conclude that C is an independent set.
- For $u \in B$ there are three cases:
  - For every k $f^k(u) \in B$ and $f^{n-1}(u) = f^n(u)$ for some n.
  - For every k $f^k(u) \in B$ and $f^{n-1}(u) \neq f^n(u)$ for every n.
  - There exists n s.t. $f^n(u) \in C$.
- In first and third case we define $c(u) = n$ modulo 2, $c(u) = 2$ if $u \in C$. The second case is imposible so we finish the proof.

Final words

- There is a Borel space V and Borel function $F : V \to V$ s.t. $\chi_B(G_F) = \omega$.

Final words

- There is a Borel space V and Borel function $F : V \to V$ s.t. $\chi_B(G_F) = \omega$.

- We can define $G_{\{f_i : i \in n\}}$ and then $\chi(G_{\{f_i : i \in n\}}) \leq 2n + 1$. Is it true that $\chi_B(G_{\{f_i : i \in n\}}) \leq 2n + 1$ or $\chi_B(G_{\{f_i : i \in n\}}) = \omega$ for Borel functions?

Final words

- There is a Borel space V and Borel function $F : V \to V$ s.t. $\chi_B(G_F) = \omega$.

- We can define $G_{\{f_i : i \in n\}}$ and then $\chi(G_{\{f_i : i \in n\}}) \leq 2n + 1$. Is it true that $\chi_B(G_{\{f_i : i \in n\}}) \leq 2n + 1$ or $\chi_B(G_{\{f_i : i \in n\}}) = \omega$ for Borel functions?

- There is an analoguous definition for coloring of edges and Vizing's theorem says that whenever every vertex has at most n degree then $\chi'(G) \in \{n, n + 1\}$. Is it true for Borel colorings?

$\mathcal{GRACIAS}$.
$\mathcal{THANK\ \ YOU}$
$\mathcal{DEKUJI}$