Winter School in Abstract Analysis, section Set Theory

ABOUT THE REAPING NUMBER OF DENSE SUBSETS OF THE RATIONALS

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Combinatorics of dense subsets of the rationals, B. Balcar, M. Hrušák and F.Hernández-Hernández

- The main object of study of this paper is the partial order $(\mathit{Dense}(\mathbb{Q}), \subseteq_{\mathsf{nwd}})$.
- Among other intersting results, they formulate cardinal invariants analogous to the ones that appear in Van Dowen's Diagram, and prove several relations between them.

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- In some cases, these cardinal invariants coincide with the corresponding version in Van Dowen's Diagram.

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Definition

A family $\mathcal{R} \subseteq Dense(\mathbb{Q})$ is a *dense*-reaping family provided that for any $X \in Dense(\mathbb{Q})$, there is $Y \in \mathcal{R}$ such that $Y \setminus X \notin Dense(\mathbb{Q})$ or $X \cap Y \notin Dense(\mathbb{Q})$.

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The *dense-reaping* number $\mathfrak{r}_{\mathbb{Q}}$ is defined as the minimum cardinality of a dense-reaping family, i.e.,

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Theorem(Balcar, Hrušák, Hernández-Hernández).

The following holds:

- $\mathfrak{r}_{\mathbb{Q}} = \mathfrak{r}(\mathcal{P}(\mathbb{Q})/\text{nwd})$.
- $\max\{\mathfrak{r}, cof(\mathcal{M})\} \leq \mathfrak{r}_{\mathbb{Q}} \leq \mathfrak{i}$.

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- Does $\mathcal{P}(\mathbb{Q})/\text{nwd}$ collapse $\mathfrak{c}$ to $\mathfrak{h}_{\mathbb{Q}}$? Yes (D. Carolina Montoya, J. Brendle)
- Are the following relatively consistent with ZFC?:
  - $\mathfrak{h} < \mathfrak{h}_{\mathbb{Q}}$ Yes (Brendle).
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Main Theorem

The inequality $\mathfrak{r}_{\mathbb{Q}} < \mathfrak{i}$ is relatively consistent with ZFC

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Remember that an ideal $\mathscr{I}$ on ω is saturated if the quotient $\mathcal{P}(\omega)/\mathscr{I}$ has the c.c.c.

There are several forcing notions satisfying the following theorem, but we are using the one in $Con(\mathfrak{i} < \mathfrak{u})$.

Theorem (S. Shelah).

Let $\mathscr{I}$ be a saturated ideal. Then there is a forcing notion $\mathcal{Q}_{\mathscr{I}}$ such that

- $\mathcal{Q}_{\mathscr{I}}$ is proper and ω^ω-bounding.
- $\mathcal{Q}_{\mathscr{I}}$ adds a set $\dot{X}$ such that for any $Y \in \mathscr{I}^+ \cap V$, $\mathcal{Q}_{\mathscr{I}} \Vdash |\dot{X} \cap Y| = |Y \setminus \dot{X}| = \omega$.

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Lemma.

For every maximal independent family $\mathcal{J}$, there is a saturated ideal $\mathscr{I}$ such that the forcing $\mathcal{Q}_{\mathscr{I}}$ forces that $\mathcal{J}$ is not longer a maximal independent family.

So making an CSI of length ω_2 of forcings $\mathcal{Q}_{\mathscr{I}}$, where every saturated ideal is destroyed (via a bookkeeping device), we get a model where $\mathfrak{i}$ is big.

We still have to preserve the family *Dense*$(\mathbb{Q})$ from the ground model as a dense-reaping family. How?

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Definition.

A filter $\mathcal{U} \subseteq \mathit{Dense}(\mathbb{Q})$ is called *selective* $\mathbb{Q}$-filter, whenever it is a p-filter and a q-filter.
A $\mathbb{Q}$-filter $\mathcal{U}$ is maximal if it is maximal relative to $\mathit{Dense}(\mathbb{Q})$.

Let $\mathscr{I}$ be an ideal on ω. A function $f : \omega \to \omega$ is $\mathscr{I}$-to-one if the for all $n \in \omega$ $f^{-1}(n) \in \mathscr{I}$.

A filter $\mathcal{U}$ is good for $\mathscr{I}$ if for NO $\mathscr{I}$-to-one function f, $f^*(\mathscr{I}^*) \cup \mathcal{U}$ generates a filter.

Theorem

Assume $\mathscr{I}$ is a saturated ideal, and let $\mathcal{U}$ be a maximal selective $\mathbb{Q}$-filter good for $\mathscr{I}$. Then $\mathcal{Q}_{\mathscr{I}}$ forces that $\mathcal{U}$ generates a maximal selective $\mathbb{Q}$-filter.

In other words, if $\mathcal{Q}_{\mathscr{I}}$ destroys a maximal selective $\mathbb{Q}$-filter $\mathcal{U}$, it is because $\mathcal{U}$ is not good for the ideal $\mathscr{I}$, i.e, there is a function $\mathscr{I}$-to-one such that $f^*(\mathscr{I}^*) \cup \mathcal{U}$ generates a filter.

From here on, whenever an ideal is mentioned, it will be supposed to be a saturated ideal.

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Lemma(GCH).

There is a family $\mathscr{F}$ of maximal selective $\mathbb{Q}$-filters such that:

- $\mathscr{F}$ has cardinality ω_2.
- For every saturated ideal $\mathscr{I}$, the family $\{\mathcal{U} \in \mathscr{F} : \mathcal{U}$ is not good for $\mathscr{I}\}$ is countable. In other words, all but countably many filters in $\mathscr{F}$ are good for $\mathscr{I}$.
- Moreover, the above property is preserved in forcing extensions preserving ω_1.

Note that if we start with a model of GCH, and $\mathscr{F}$ is the family of the above lemma, then whenever we force with $\mathcal{Q}_{\mathscr{I}}$, there are ω_2 maximal selective $\mathbb{Q}$-filters from the ground model that survives as maximal selective $\mathbb{Q}$-filters.

The same is true for finite iterations where the iterands are of the form $\mathcal{Q}_{\mathscr{I}}$.

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- For every saturated ideal $\mathscr{I}$, the family $\{\mathcal{U} \in \mathscr{F} : \mathcal{U}$ is not good for $\mathscr{I}\}$ is countable. In other words, all but countably many filters in $\mathscr{F}$ are good for $\mathscr{I}$.
- Moreover, the above property is preserved in forcing extensions preserving ω_1.

Note that if we start with a model of GCH, and $\mathscr{F}$ is the family of the above lemma, then whenever we force with $\mathcal{Q}_{\mathscr{I}}$, there are ω_2 maximal selective $\mathbb{Q}$-filters from the ground model that survives as maximal selective $\mathbb{Q}$-filters.

The same is true for finite iterations where the iterands are of the form $\mathcal{Q}_{\mathscr{I}}$.

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Lemma.

Let $\mathcal{U}$ be a maximal selective $\mathbb{Q}$-filter. Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ be a countable support iteration such that for all $\beta < \alpha$, $\mathbb{P}_\beta$ preserves $\mathcal{U}$ and $\mathbb{P}_\beta \Vdash \dot{\mathbb{Q}}_\beta$ *is proper*. Then $\mathbb{P}_\alpha$ preserves $\mathcal{U}$ as a maximal selective $\mathbb{Q}$-filter.

You can derived this as a corollary from a more general theorem of Shelah (In *Con*$(\mathfrak{i} < \mathfrak{u})$, the last lemma).

This together with the previous lemma implies that if $\mathbb{P}$ is a CSI of forcings of the form $\mathcal{Q}_{\mathscr{I}}$, then in every step of the iteration there are ω_2 maximal $\mathbb{Q}$-filters in $\mathscr{F}$.

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Putting all together...

- Start with a model of GCH and let $\mathscr{F}$ be the family of the above lemma.
- Make a CSI of length ω_2 such that every succesor step of the iteration has the form $\mathcal{Q}_{\mathscr{I}}$.
- This raise up the cardinal invariant $\mathfrak{i}$.
- Every step of the iteration destroys at most ω_1 maximal selective $\mathbb{Q}$-filters in the family $\mathscr{F}$, so in every step of the iteration there are ω_2 maximal selective $\mathbb{Q}$-filters from the ground model which survive as maximal selective $\mathbb{Q}$-filters.
- This implies that every dense subset of $\mathbb{Q}$ is reaped by some $X \in Dense(\mathbb{Q}) \cap V$, that is, $\mathfrak{r}_{\mathbb{Q}} = \omega_1$.

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Thank you very much!!