Some Remarks on Weak Diamonds

Michael Hrušák, Osvaldo Guzmán González

C.C.M, Morelia

Introduction

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- It turns out that $\Diamond$ is quite a strong axiom, it implies the existence of a Suslin tree, the existence of an Ostaszewski space and the Continuum Hypothesis (CH).

- It is interesting to find weaker "$\Diamond$ like principles", in particular, we are interested in guessing principles compatible with the failure of some cardinal arithmetic assumptions.

Mirna Džamonja, Michael Hrušák and Justin Tatch Moore introduced the "parametrized diamonds" which have the same relation to $\Diamond$ as the cardinal invariants of the continuum have to the Continuum Hypothesis. We want to generalize their work to bigger cardinals. Recall the definition of "weak diamond" introduced by Devlin and Shelah,

$\Phi_\kappa(2,=)$ For every coloring $F : 2^{<\kappa} \longrightarrow 2$ there is a $g : \kappa \longrightarrow 2$ such that for every $R \in {}^{\omega_1}2$ the set $\{\alpha \mid C(R \restriction \alpha) = g(\alpha)\}$ is stationary.

It turns out that $\Phi_\kappa(2,=)$ is a consequence of CH. In fact, $\Phi_{\omega_1}(2,=)$ is really a cardinal arithmetic assumption,

Theorem (Devlin, Shelah)

For every cardinal κ, the weak diamond $\Phi_{\kappa^+}(2,=)$ is equivalent to $2^\kappa < 2^{\kappa^+}$.

This makes $\Phi_\kappa(2,=)$ a little too strong for some applications.

However, it turns out that for most applications of $\Phi_\kappa(2,=)$ the colorings are "nicely definable".

Definition (Chang)

Given a cardinal μ, we define $L(OR^\mu)$ as the smallest transitive ZF model containing all ordinals and closed under taking sequences of size μ.

Given $\mu \leq \kappa$ we define the principles,

$\diamondsuit_\kappa^\mu(2,=)$ For every coloring $F : 2^{<\kappa} \longrightarrow 2$ such that $F \restriction 2^\alpha \in L(OR^\mu)$ for every $\alpha < \kappa$ there is a $g : \kappa \longrightarrow 2$ such that for every $R \in {}^{\omega_1}2$ the set $\{\alpha \mid C(R \restriction \alpha) = g(\alpha)\}$ is stationary.

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The cases where $\kappa = \omega_1$ and $\mu = \omega$ were (with a slight modification) concider by Mirna Džamonja, Michael Hrušák and Justin Tatch Moore.

Theorem

If $\mathfrak{c} \geq \omega_2$ and $\diamondsuit_{\omega_2}^{\omega_2}(2, =)$ is true, then there are two non isomorphic ω_2-dense sets of the reals (In fact, for every ω_2-dense set $A \subseteq \mathbb{R}$ there is a ω_2-dense set $B \subseteq A$ such that A and B are non isomorphic.

Theorem

If $\mathfrak{b}(\kappa) = \kappa^+$ and $\diamondsuit_{\kappa^+}^{\kappa^+}(\kappa^+, =)$ then $\mathfrak{a}(\kappa^+) = \kappa^+$.

Recall the result of Devlin and Shelah,

Theorem (Devlin, Shelah)

For every cardinal κ, the weak diamond $\Phi_{\kappa^+}(2, =)$ is equivalent to $2^\kappa < 2^{\kappa^+}$.

Although it may not look like, it turns out that $\diamondsuit_{\kappa^+}^{\kappa^+}(2, =)$ is also a cardinal arithmetic assumption... but for a different universe than ours.

Given two sets a and b, we say $a \precsim b$ if there is a surjection from b to a.

Theorem

For every cardinal κ, the diamond $\diamondsuit_{\kappa^+}^{\kappa^+}(2, =)$ is equivalent to $L\left(OR^{\kappa^+}\right) \models 2^{\kappa^+} \not\precsim 2^\kappa$.

How to construct a surjection from 2^{ω_1} into 2^{ω_2} just using a sequence of lenght ω_2? There are many ways, here is one.

Definition

We say an almost disjoint family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *normal* if for every $\mathcal{B} \subseteq \mathcal{A}$ there is $X \subseteq \omega$ such that,

1. $B \subseteq^* X$ for every $B \in \mathcal{B}$.
2. $C \cap X =^* \emptyset$ for every $C \in \mathcal{A} - \mathcal{B}$.

Theorem

MA $+\ \omega_2 < \mathfrak{c}$ *implies* $\neg\Diamond^{\omega_2}_{\omega_2}(2, =)$.

Demostración.

Under MA$+\ \omega_2 < \mathfrak{c}$ there are normal AD families and with one of them we can build a surjection from 2^{ω} onto 2^{ω_2}. □

Just one little remark regarding the different notions of definibility. The principles $\diamondsuit_{\omega_1}^{\omega}(2,=)$ and $\diamondsuit_{\omega_1}^{\omega_1}(2,=)$ look similar however they are not the same,

Theorem

A Suslin tree forces $\diamondsuit_{\omega_1}^{\omega}(2,=)$ but $\diamondsuit_{\omega_1}^{\omega_1}(2,=)$ may fail after the extention.

Thank You!

Thank you for your attention!
Time for lunch!