Cross-like constructions and refinements

Daniel Soukup

Introduction

First ideas:

- Sorgenfrey line: convergence $\leftrightarrow$ convergence from the right,
- generalize this: convergence $\leftrightarrow$ convergence from given directions.
- What topologies capture this property?

Classical constructions:

- the **cross-topology on $\mathbb{R}^2$**,
- the **radiolar-topology on $\mathbb{R}^2$**,
- the **cross-topology on $X \times Y$**.

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Results and concerns from the literature

Investigating basic topological properties (late 60's):

- separation axioms: the usual failure of T_3,
- density, covering properties,
- connectivity.

The question of regularity:

- regularizations, hard-to-see topologies (Knight-Moran-Pym (1968)),
- conditions which ensure regularity, normality for the cross-topology on $X \times Y$. (Hart-Kunen (2002))

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A general idea of refining (notion from R. Brown):

- take a space (X, τ),
- take any family $\mathcal{E}$ of subsets of X,
- let $\tau_{\mathcal{E}} = \{U \subseteq X : U \cap E$ is relatively open in E for all $E \in \mathcal{E}\}$.

Let $S \subseteq S^1$ be a set of unit vectors, usually called *directions*.

Definition

The set $U \subseteq \mathbb{R}^2$ is S-radiolar open iff for every $x \in U$ and every $s \in S$ there is a line segment in direction s in U which contains x. Let $\mathcal{R}(S)$ denote this topology.

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Basics

Two special and two trivial cases:

- cross topology,
- for $S = S^1$ the radiolar topology,
- for $S = \{s\}$ $\mathcal{R}(S)$ is the disjoint union of $\mathfrak{c}$-many Sorgenfrey lines,
- for $S = \{s, -s\}$ $\mathcal{R}(S)$ is the disjoint union of $\mathfrak{c}$-many Euclidean lines.

Simple properties for a nontrivial $\mathcal{R}(S)$ space:

- What is the convergence we have?
- The $\mathcal{R}(S)$ spaces are **Hausdorff and separable**.
- There always exists a **closed discrete subset in $\mathcal{R}(S)$ of cardinality $\mathfrak{c}$**, hence these spaces are non normal, non Lindelöf, non hereditaraly separable.

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The character of $\mathcal{R}(S)$

By generalizing a proof of Hart and Kunen:

Theorem

For nontrivial $S \subseteq S^1$ we have $\chi(\mathcal{R}(S)) = 2^{\mathfrak{c}}$.

Thus $\mathcal{R}(S)$ **is non regular** since for any regular space X: $w(X) \leq 2^{d(X)}$.

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A set $S \subseteq S^1$ is splayed iff it cannot be covered by a closed half circle, S contains a full direction is there is a $s \in S^1$ such that $\{s, -s\} \subseteq S$.

Theorem

- *$\mathcal{R}(S)$ is connected iff S is splayed.*
- *There exists an uncountable compact subspace in $\mathcal{R}(S)$ iff there is a full direction in S.*
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Proposition

For ***symmetric and non-symmetric*** *S sets the corresponding $\mathcal{R}(S)$ topologies are* ***non homeomorphic****.*

Definition

A space X is ***symmetrizable*** *iff there is a* ***symmetric*** *$d : X \times X \to \mathbb{R}$ on X:*

1. *for all $x, y \in X : d(x, y) = d(y, x) \geq 0$,*
2. *$d(x, y) = 0 \Leftrightarrow x = y$,*

such that $U \subseteq X$ is open iff for any $x \in U$ there is a $\epsilon > 0$ such that $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \subseteq U$.

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*A space X is **symmetrizable** iff there is a **symmetric** $d : X \times X \to \mathbb{R}$ on X:*

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such that $U \subseteq X$ is open iff for any $x \in U$ there is a $\epsilon > 0$ such that $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \subseteq U$.

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Definition (Arhangelskii)

For a space (X, τ) and a point $x \in X$ a family of closed sets is a **weak base at x** *iff*

- $x \in \bigcap \mathcal{B}$,
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Let the **weak base character** be $\chi_w(x, X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a weak base at $x\}$ and $\chi_w(X) = \sup\{\chi_w(x, X) : x \in X\}$.

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Can one give a nice characterization of pairs $S, T \subseteq S^1$ such that the corresponding $\mathcal{R}(S)$ and $\mathcal{R}(T)$ topologies are homeomorphic?

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Defining the $\mathcal{B}(S)$ spaces
A way of regularization

The butterfly-construction:

- Let (M, τ) be a metric space.
- (M, τ^*) is a *butterfly-space* over (M, τ) if every point has a base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ $B \setminus \{x\} \in \tau$.

Definition

Fix a $S \subseteq S^1$, $x \in \mathbb{R}^2, r > 0$. Let us use the following notion:

$$\textbf{S(x,r)} = \bigcup \{[x, x + rs) : s \in S\}.$$

Definition

*The $\mathcal{B}(S)$ **topology** is defined as follows: an $U \subseteq \mathbb{R}^2$ is said to be $\mathcal{B}(S)$-open iff for every point $x \in U$ there is a $x \in V \subseteq U$ such that $S(x, r) \subseteq V$ for some $r > 0$ and $V \setminus \{x\}$ is Euclidean open.*

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For $S = S^1$ $\mathcal{B}(S^1)$ is the Euclidean topology.

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Suppose that ***there is no missing full direction in S****. Then for every open set G in $\mathcal{B}(S)$,* ***G and its Euclidean interior can only differ in $\aleph_0$ many points.***

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Trivially: if there is a missing full direction $\Rightarrow$ there is a closed discrete subset of size $\mathfrak{c}$.

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How one can differentiate $\mathcal{B}(S)$ topologies?

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