Ideals related to Laver and Miller trees

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2. if $t \supseteq s$, then $\mathrm{br}_p(t) = \{\alpha \in \kappa : t^\frown\langle\alpha\rangle \in p\} \in \mathcal{I}^+$.

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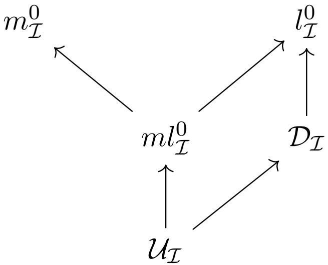

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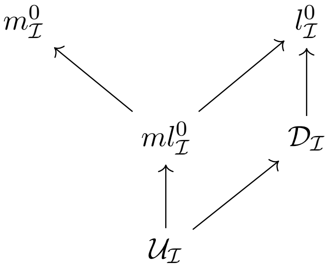

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If $2^\kappa = \mathfrak{c}$, the ideal $\mathcal{I}$ is not prime and $\mathrm{non}(ml^0_{\mathcal{I}}) = \mathfrak{c}$, then there is a set $A \subseteq {}^\omega\kappa$ such that $A \in ml^0_{\mathcal{I}}$ and $A \notin \mathcal{D}_{\mathcal{I}}$. Hence $\mathcal{U}_{\mathcal{I}} \subsetneq ml^0_{\mathcal{I}}$ and $\mathcal{D}_{\mathcal{I}} \subsetneq l^0_{\mathcal{I}}$.

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An ideal $\mathcal{I}$ is *locally prime*, iff the set

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Thank you for your attention!

References

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