Isomorphisms between corona algebras of the form $\prod_n M_{k(n)}(\mathbb{C})/\bigoplus_n M_{k(n)}(\mathbb{C})$

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Introduction

- Fix a separable Hilbert space H. Let $B(H)$ denote the space of all bounded linear operators on H and $\mathcal{K}(H)$ be the closed ideal of all compact operators.
- A C*-algebra is a Banach algebra with involution which is *-isomorphic to a subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.
- **Example.** For a locally compact Hausdorff space X, the algebra

$$C_0(X) = \{f : X \to \mathbb{C} : f \text{ is continuous and vanishes at infinity}\}$$

is a C*-algebra with the involution $f^* = \bar{f}$, pointwise multiplication and the sup norm. If X is compact we write $C(X) = C_0(X)$.

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Unitizations of C*-algebras

Fact

By the Gelfand-Naimark duality every commutative C-algebra is isometrically *-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space X.*

- For a non-unital C*-algebra $\mathcal{A}$ there are various ways in which $\mathcal{A}$ can be embedded as an ideal in a unital C*-algebra. If $\mathcal{A} = C_0(X)$ is commutative this corresponds to the ways in which the locally compact Hausdorff space X can be embedded as an open set in a compact Hausdorff space Y.
- The minimal way to do so, is to take the one-point compactification of X and the maixmal way is the Čech-Stone compactification βX of X. The C*-analog of the Čech-Stone compactification is called the *multiplier algebra* of $\mathcal{A}$.

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Multiplier Algebras

- A (two-sided and closed) ideal of $\mathcal{A}$ is called *essential* if it has a non-trivial intersection with any non-zero ideal of $\mathcal{A}$.
- The multiplier algebra $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ is the unital C*-algebra containing $\mathcal{A}$ as an essential ideal, and is universal in the sense that whenever $\mathcal{A}$ is an ideal of a unital C*-algebra $\mathcal{D}$, the identity map on $\mathcal{A}$ extends uniquely to a *-homomorphism from $\mathcal{D}$ into $\mathcal{M}(\mathcal{A})$.

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Multiplier Algebras

- There are several ways of constructing $\mathcal{M}(\mathcal{A})$. A traditional way is to take a faithful non-degenerate representation ρ of $\mathcal{A}$ on a Hilbert space H, and consider $\mathcal{M}(\mathcal{A})$ as the idealizer of $\rho(\mathcal{A})$ in $B(H)$,

$$\mathcal{M}(\mathcal{A}) \cong \{m \in B(H) \ : \ \forall a \in \mathcal{A} \quad m\rho(a) \text{ and } \rho(a)m \in \rho[\mathcal{A}]\}.$$

Definition

The quotient C*-algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is called the *corona* of $\mathcal{A}$ and is denoted by $\mathcal{C}(\mathcal{A})$.

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Corona Algebras

Examples of corona algebras.

1. If $\mathcal{A}$ is unital, then $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. Therefore the corona of $\mathcal{A}$ is trivial.
2. If $\mathcal{A} = \mathcal{K}(H)$, the closed ideal of all compact operators on a Hilbert space H, then $\mathcal{M}(\mathcal{A}) = \mathcal{B}(H)$. The corona of $\mathcal{A}$ is the Calkin algebra $\mathcal{C}(H)$.
3. If X is a locally compact Hausdorff space, then

$$\mathcal{M}(C_0(X)) \cong C(\beta X)$$
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Examples

4. Suppose $\mathcal{A}_n$ is a sequence of C*-algebras and let

$$\prod_n^{\infty} \mathcal{A}_n = \{(x_n) : \quad x_n \in \mathcal{A}_n \ \text{ and } \ \sum_{n=1}^{\infty} \|x_n\| < \infty\}$$

$$\bigoplus_n^{\infty} \mathcal{A}_n = \{(x_n) \in \prod_n \mathcal{A}_n : \|x_n\| \to 0\}.$$

If $\mathcal{B} = \bigoplus_n \mathcal{A}_n$ then $\mathcal{M}(\mathcal{B}) = \prod_n \mathcal{A}_n$. The corona of $\mathcal{B}$ is $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$.

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General reduced products

Assume $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $\mathcal{A}_n$ is a sequence of C*-algebras. Then

$$\bigoplus_{\mathcal{I}} \mathcal{A}_n = \{(x_n) \in \prod_n \mathcal{A}_n : \limsup_{\mathcal{I}} \|x_n\| = 0\}$$

is a closed ideal of $\prod_n \mathcal{A}_n$. The quotient C*-algebra is called the *reduced product* of the sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ over the ideal $\mathcal{I}$. Clearly if $\mathcal{I} = Fin$, the reduced product $\prod \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$ is the corona algebra of $\bigoplus_n \mathcal{A}_n$.

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- Let M_n denote the space of all $n \times n$ matrices over $\mathbb{C}$ and assume $\{k(n) : n \in \mathbb{N}\}$ is a sequence of natural numbers. Then the quotient $\prod M_{k(n)} / \bigoplus M_{k(n)}$ is an important example of corona algebras.
- Note that $\prod M_{k(n)} / \bigoplus M_{k(n)}$ can be considered as a non-commutative analogous of ℓ_∞ / c_o.

Our goal is to study the automorphisms of the corona algebra $\prod M_{k(n)} / \bigoplus M_{k(n)}$, or more generally the isomorphisms between $\prod M_{k(n)} / \bigoplus_{\mathcal{I}} M_{k(n)}$ and $\prod M_{l(n)} / \bigoplus_{\mathcal{J}} M_{l(n)}$, for definable ideals $\mathcal{I}$, $\mathcal{J}$ and sequences of natural numbers $\{k(n)\}$ and $\{l(n)\}$.

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Trivial homomorphisms

Definition

In general for quotient structures $X/\mathcal{I}$, $Y/\mathcal{J}$ and Φ a homomorphism between them, Φ is called *trivial* if there is a homomorphism lifting, i.e., there is $\Phi_* : X \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "\Phi_*"] \arrow[d, "\pi_{\mathcal{I}}"'] & Y \arrow[d, "\pi_{\mathcal{J}}"] \\
X/\mathcal{I} \arrow[r, "\Phi"'] & Y/\mathcal{I}
\end{tikzcd}$$

commutes, where $\pi_{\mathcal{I}}$ and $\pi_{\mathcal{J}}$ are the canonical quotient maps.

Motivation

The following theorem has been the main motivation for the results of this talk.

Theorem (Farah, Shelah, 2014)

Assume there exists a measurable cardinal and let $\mathcal{I}$ and $\mathcal{J}$ be Borel ideals on ω. There is a forcing extension in which every isomorphism between quotient Boolean algebras $P(\omega)/\mathcal{I}$ and $P(\omega)/\mathcal{J}$ has a continuous lifting.

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Trivial automorphisms

Definition

Assume $\mathcal{A}$ is a unital C*-algebras. Each unitary $u \in \mathcal{A}$ ($uu^* = u^*u = 1$) defines an automorphism Ad_u of $\mathcal{A}$ given by $a \to uau^*$ and such automorphisms are called *inner automorphisms*.

Theorem (Phillips, Weaver)

If the CH holds, the Calkin algebra has an outer (not inner) automorphism.

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Ideals $\mathcal{I}$ and $\mathcal{J}$ are Rudin-Keisler isomorphic, $\mathcal{I} \approx_{RK} \mathcal{J}$, if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection $\sigma : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ such that for all $X \subseteq \mathbb{N} \setminus A$ we have

$$X \in \mathcal{I} \quad \leftrightarrow \quad \sigma[X] \in \mathcal{J}.$$

Definition

An ideal $\mathcal{I}$ is a P-ideal if the partial ordering $\langle \mathcal{I}, \subseteq^* \rangle$ is σ-directed.

Fact (Solecki)

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Trivial isomorphisms

Assume ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ are Rudin-Keisler isomorphic via a bijection $\sigma : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ for $A \in \mathcal{I}$ and $B \in \mathcal{J}$. If $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ are sequences of C*-algebras such that there are isomorphisms $\varphi_n : \mathcal{A}_n \cong \mathcal{B}_{\sigma(n)}$ for every $n \in \mathbb{N} \setminus A$, then there is an obvious (and trivial) isomorphism Φ between algebras $\prod_n \mathcal{A}_n / \bigoplus_{\mathcal{I}} \mathcal{A}_n$ and $\prod_n \mathcal{B}_n / \bigoplus_{\mathcal{J}} \mathcal{B}_n$.
Namely define Φ by

$$\Phi(\pi_{\mathcal{I}}((a_n))) = \pi_{\mathcal{J}}(\varphi_n(a_n)).$$

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Theorem (G. 2014)

It is relatively consistent with ZFC that for analytic P-ideals $\mathcal{I}$, $\mathcal{J}$ on $\mathbb{N}$, the reduced products $\prod M_{k(n)}/\bigoplus_{\mathcal{I}} M_{k(n)}$ and $\prod M_{l(n)}/\bigoplus_{\mathcal{J}} M_{l(n)}$ are isomorphic if and only if

1. $\mathcal{I} \approx_{RK} \mathcal{J}$, *i.e., there are sets* $A \in \mathcal{I}$, $B \in \mathcal{J}$, *and a bijection* $\nu : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ *such that for all* $X \subseteq \mathbb{N} \setminus A$ *we have*

$$X \in \mathcal{I} \quad \leftrightarrow \quad h[X] \in \mathcal{J}$$

2. *and* $k(\nu(n)) = l(n)$ *for all* $n \in \mathbb{N} \setminus A$.

Moreover if Φ *is such an isomorphism, there exists an isometry* $u : \prod_{n \in \mathbb{N} \setminus A} M_{k(n)} \to \prod_{n \in \mathbb{N}} M_{l(n)}$ *such that* $\Phi(\pi_{\mathcal{I}}(a)) = \pi_{\mathcal{J}}(uau^*)$.

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Corollary

Assume there is a measurable cardinal. There is a forcing extension in which every automorphism of $\prod M_{k(n)}/\bigoplus M_{k(n)}$ is inner.

Some remarks.

- In particular any isomorphism $\Phi : \prod M_{k(n)}/\bigoplus_{\mathcal{I}} M_{k(n)} \to \prod M_{l(n)}/\bigoplus_{\mathcal{J}} M_{l(n)}$ has a *-homomorphism lifting $\Psi : \prod M_{k(n)} \to \prod M_{l(n)}$ (trivial).
- In the theorem if we replace analytic P-ideals with "Borel ideals" we can still obtain continuous liftings of any such isomorphism (each $\prod_n M_{k(n)}$ is equipped with the strong operator topology).

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- In the theorem if we replace analytic P-ideals with "Borel ideals" we can still obtain continuous liftings of any such isomorphism (each $\prod_n M_{k(n)}$ is equipped with the strong operator topology).

Corollary

Assume there is a measurable cardinal. There is a forcing extension in which every automorphism of $\prod M_{k(n)}/\bigoplus M_{k(n)}$ *is inner.*

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- The Farah-Shelah's result (in its dual form) follows from this theorem, since it corresponds to situation when the maps are restricted to the centers of the algebras.
- The existence of the measureable cardinal is only to ensure that all Π^1_2 sets have the property of Baire.
- The forcing notion used is a countable support iteration of length $\mathfrak{c}$ of the random forcing and *groupwise Silver forcings*. These forcing notions are of the form $\mathbb{P}_{\mathcal{I}} = Bor(\mathbb{R})/\mathcal{I}$, for a σ-ideal $\mathcal{I}$. These forcing notions and their countable supports iterations are well-studied by J. Zapletal.

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Groupwise Silver forcing posets

- Let $\vec{I} = (I_n)$ be a partition of $\mathbb{N}$ into non-empty finite intervals and G_j be a finite set for each $j \in \mathbb{N}$. For each n define $F_n^{\vec{I}} = \prod_{i \in I_n} G_i$ and let $F^{\vec{I}} = \prod_{n \in \omega} F_n$ endowed with the product topology.
- Define $\mathbb{S}_{F^{\vec{I}}}$ the groupwise Silver forcing associated to $F^{\vec{I}}$, to be the following forcing notion: A condition $p \in \mathbb{S}_{F^{\vec{I}}}$ is a function from $M \subset \mathbb{N}$ into $\bigcup_{n=0}^{\infty} F_n$, such that the complement of M is infinite and $p(n) \in F_n$. A condition p is stronger than q if p extends q.

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Model theory and presence of CH

In the last few years the model theory for operator algebras has been developed and specialized from the model theory for metric structures. This has proved to be very fruitful as many properties of C*-algebras and tracial von Neumann algebras have equivalent model theoretic reformulations.

Theorem (Farah-Shelah, 2014)

If $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is a sequence of C-algebras, then the corona algebra $\prod \mathcal{A}_n / \bigoplus \mathcal{A}_n$ is countably saturated.*
[B. Hart] Hence assuming CH if each $\mathcal{A}_n$ is separable, $\prod \mathcal{A}_n / \bigoplus \mathcal{A}_n$ has $2^{\aleph_1}$ many automorphisms. In particular, it has outer automorphisms.

Corollary

The assertion that all automorphisms of the corona algebra $\prod_n \mathrm{M}_{k(n)}(\mathbb{C}) / \bigoplus_n \mathrm{M}_{k(n)}(\mathbb{C})$ are trivial, is independent from ZFC.

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Non-trivial isomorphisms

Fact

Assuming CH, $C(\beta\omega \setminus \omega) \cong C(\beta\omega^2 \setminus \omega^2)$, since by a classical result of Parovičenko under CH, $\beta\omega \setminus \omega$ and $\beta\omega^2 \setminus \omega^2$ are homeomorphic. However under the proper forcing axiom they are not isomorphic [Dow-Hart].

I. Farah asked whether there are examples of "genuinely" non-commutative corona algebras which are non-trivially isomorphic (under CH).

One way to obtain non-trivial isomorphisms (under CH) between non-commutative coronas is by tensoring $C(\beta\omega \setminus \omega)$ and $C(\beta\omega^2 \setminus \omega^2)$ with a full matrix algebra. However, such non-trivial isomorphisms are just amplifications of the non-trivial isomorphisms between their corresponding commutative factors.

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Theorem (G. 2014)

There is an increasing sequence of natural numbers $\{k_\infty(i) : i \in \mathbb{N}\}$ such that if $\{g(i)\}$ and $\{h(i)\}$ are two subsequences of $\{k(i)\}$, then under the Continuum Hypothesis

$$\mathcal{M}_g = \prod_i M_{g(i)} / \bigoplus M_{g(i)} \cong \mathcal{M}_h = \prod_i M_{h(i)} / \bigoplus M_{h_(i)}$$

and the following are equivalent.

1. *$\mathcal{M}_g$ and $\mathcal{M}_h$ are isomorphic in ZFC.*
2. *$\mathcal{M}_g$ and $\mathcal{M}_h$ are trivially isomorphic, i.e., $\{g(i) : i \in \mathbb{N}\}$ and $\{h(i) : i \in \mathbb{N}\}$ are equal modulo finite sets.*

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A metric Feferman-Vaught theorem

The proof of the previous theorem relies on the following strong tool.

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There is an extension of the classical Feferman-Vaught theorem to reduced products of metric structures.

In the classical model theory Feferman-Vaught theorem recursively determines the truth value of a formula φ in reduced products of discrete structures $\{\mathcal{A}_n : n \in \mathbb{N}\}$ over an ideal $\mathcal{I}$ on $\mathbb{N}$, by the truth values of certain formulas in the models $\mathcal{A}_n$ and in the Boolean algebra $P(\mathbb{N})/\mathcal{I}$.

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Thank You