Looking for differences

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Find the difference

$$\mathcal{D}'\mathcal{Q}\acute{\mathcal{S}}$$

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Difference

$$\acute{\mathcal{S}} \cap \mathcal{Q} \setminus \mathcal{D}$$

$$\acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$$

Plan

1. Definitions
2. Differences
3. Algebrability
4. Differences
5. What is the difference?

Definitions

Let $f\colon \mathbb{R} \to \mathbb{R}$.

- Function f is *quasi-continuous* if for all $a < x < b$ and each $\varepsilon > 0$ there exists a nondegenerate interval $J \subset (a, b)$ such that $\operatorname{diam} f[J \cup \{x\}] < \varepsilon$. The family of all quasi-continuous functions will be denoted by $\mathcal{Q}$.
- Function f is *Świątkowski* if for all $a < b$ with $f(a) \neq f(b)$, there is a y between $f(a)$ and $f(b)$ and an $x \in (a, b) \cap \mathcal{C}(f)$ such that $f(x) = y$. The family of all Świątkowski functions will be denoted by $\acute{\mathcal{S}}$.
- The family of all Darboux functions will be denoted by $\mathcal{D}$.

Differences

$\acute{\mathcal{S}}\mathcal{Q} \setminus \mathcal{D}$

$$f(x) = \begin{cases} x+1, & \text{when } x \geqslant 0 \\ x-1, & \text{when } x < 0. \end{cases}$$

$\acute{\mathcal{S}} \setminus (\mathcal{Q} \cup \mathcal{D})$

$$f(x) = \begin{cases} x+1, & \text{when } x > 0, \\ 0, & \text{when} x = 0, \\ x-1, & \text{when } x < 0. \end{cases}$$

Algebraization

Definition (Aron, Gurariy, Seoane- Sepulveda, 2005)

Let $\mathcal{L}$ be a linear commutative algebra and a set $A \subset \mathcal{L}$. We say that A is κ-algebrable if $A \cup \{\theta\} \subset \mathcal{L}$ contains a κ-generated algebra B (i.e. the minimal cardinality of the set of generators of B is equal to κ).

Definition (Bartoszewicz, Głąb, 2013)

Let $\mathcal{L}$ be a commutative algebra and a set $A \subset \mathcal{L}$. We say that A is strongly κ-algebrable if $A \cup \{\theta\}$ contains a κ-generated algebra that is isomorphic to a free algebra.

Proposition

The family $\acute{\mathcal{S}}$ is at most $\mathfrak{c}$-algebrable.

Proof

By contradiction:

- take a base B of a $\mathfrak{c}^+$-dimensional vector space W contained in $\acute{\mathcal{S}}$
- there is a G_δ set A and $\mathfrak{c}^+$ Świątkowski functions $g_\alpha \in B$ such that $C(g_\alpha) = A$
- there is $\mathfrak{c}^+$ functions $g_{\alpha'} \in B$ such that $g_{\alpha'_1}(x) = g_{\alpha'_2}(x)$ for $x \in \mathcal{C}(g_{\alpha'_1}) = \mathcal{C}(g_{\alpha'_2}) = A$
- the function $f = g_{\alpha'_1} - g_{\alpha'_2}$ in not a Świątkowski function $\mathcal{C}(f) \subset f^{-1}[\{0\}]$

Algebraization

Definition

We say that a function $f : \mathbb{R} \to \mathbb{R}$ *is exponential-like (of the rank* m*) if*

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for $\beta_i \in \mathbb{R} \setminus \{0\}$, $\beta_i \neq \beta_j$ *for* $i \neq j$ *and* $a_i \in \mathbb{R} \setminus \{0\}$, $i \in \{1, 2, \ldots, m\}$.

The family of all exponential-like functions of rank m will be denoted by $\mathcal{E}_m$ and $\mathcal{E} := \bigcup_{m \in \mathbb{N}} \mathcal{E}_m$.

Algebraization

Lemma (Balcerzak, Bartoszewicz, Filipczak, 2013)

For every positive integer m, any $f \in \mathcal{E}_m$ and each $c \in \mathbb{R}$, the preimage $f^{-1}[\{c\}]$ has at most m elements. In particular there exists a finite decomposition of $\mathbb{R}$ into intervals, such that f is strictly monotone on each of them.

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Theorem (Balcerzak, Bartoszewicz, Filipczak, 2013)

Given a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{\theta\}$ for every $f \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.

Świątkowski function

Function f is *Świątkowski* if for all $a < b$ with $f(a) \neq f(b)$, there is a y between $f(a)$ and $f(b)$ and an $x \in (a, b) \cap \mathcal{C}(f)$ such that $f(x) = y$.

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Cantorval-informal definition

We say that a set is a *cantorval*, provided that it is a union of the Cantor set and all the intervals removed in every second step in the construction of the Cantor set.

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Theorem

The family $\acute{\mathcal{S}} \cap \mathcal{Q} \setminus \mathcal{D}$ is strongly $\mathfrak{c}$-algebrable.

Theorem

The family $\acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$ *is not* 1*-algebrable.*

Proof

- Show that there has to be some *gap*,
- Show that for each $f \in \acute{\mathcal{S}} \setminus (\mathcal{D} \cup \mathcal{Q})$ there exists polynomial W such that $W \circ f \notin \acute{\mathcal{S}}$.

What is the difference?

It is worth noting that in proofs where we applied Theorem by M. Balcerzak, A. Bartoszewicz and M. Filipczak we used properties of exponential-like functions which are the same as properties of polynomials.

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Thank you for your attention