Patterns of stationary reflection

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Stationary reflection

Definition
Let $\beta$ be an ordinal of uncountable cofinality.

1. $S \subseteq \beta$ is stationary in $\beta$ if $S \cap C \neq \emptyset$ for every club $C \subseteq \beta$. 

Remark
If $S \subseteq S^{\lambda \kappa}$ and $S$ reflects at $\beta$, then $\text{cf}(\beta) > \kappa$. Thus, if $\kappa$ is regular and $S \subseteq S^{\kappa+\kappa}$, then $S$ does not reflect.
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3. If $S$ is a stationary subset of $\beta$, then $S$ reflects if there is $\alpha < \beta$ such that $S$ reflects at $\alpha$. 
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3. If $S$ is a stationary subset of $\beta$, then $S$ reflects if there is $\alpha < \beta$ such that $S$ reflects at $\alpha$.

4. If $\kappa$ is a cardinal of uncountable cofinality, $\text{Refl}(\kappa)$ holds if every stationary subset of $\kappa$ reflects.
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1. $S \subseteq \beta$ is *stationary in $\beta$* if $S \cap C \neq \emptyset$ for every club $C \subseteq \beta$.
2. If $S$ is a stationary subset of $\beta$ and $\alpha < \beta$ has uncountable cofinality, then $S$ *reflects at $\alpha$* if $S \cap \alpha$ is stationary in $\alpha$.
3. If $S$ is a stationary subset of $\beta$, then $S$ *reflects* if there is $\alpha < \beta$ such that $S$ reflects at $\alpha$.
4. If $\kappa$ is a cardinal of uncountable cofinality, $\text{Refl}(\kappa)$ holds if every stationary subset of $\kappa$ reflects.

If $\kappa < \lambda$ are infinite cardinals, with $\kappa$ regular, then
$$S^\lambda_\kappa = \{ \alpha < \lambda \mid \text{cf}(\alpha) = \kappa \}.$$

**Remark**
If $S \subseteq S^\lambda_\kappa$ and $S$ reflects at $\beta$, then $\text{cf}(\beta) > \kappa$. Thus, if $\kappa$ is regular and $S \subseteq S^\kappa_\kappa^+$, then $S$ does not reflect.
Classical results

**Theorem**
If $\square_\kappa$ holds, then, for every stationary $S \subseteq \kappa^+$, there is a stationary $T \subseteq S$ that does not reflect.
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If $V = L$ and $\kappa$ is a regular, uncountable cardinal, then $\text{Refl}(\kappa)$ holds iff $\kappa$ is weakly compact.
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If $\mu$ is a singular limit of supercompact cardinals, then $\text{Refl}(\mu^+)$ holds.
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If $\mu$ is a singular limit of supercompact cardinals, then $\text{Refl}(\mu^+)$ holds.

**Theorem** (Magidor)
If $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, then there is a forcing extension in which $\kappa_n = \aleph_{n+1}$ for every $n < \omega$ and $\text{Refl}(\aleph_{\omega+1})$ holds.
Square-bracket partition relations

Definition

1. If $\lambda$ is an infinite, regular cardinal and $S \subseteq \lambda$ is stationary, we say $S$ reflects at arbitrarily high cofinalities if, for every regular $\kappa < \lambda$, there is $\beta \in S^\lambda_\kappa$ such that $S$ reflects at $\beta$.
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2. If $\mu \leq \lambda$ are cardinals, then $[\lambda]^\mu = \{X \subseteq \lambda \mid |X| = \mu\}$. $[\lambda]^{<\mu}$ is defined in the obvious way.
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3. $\lambda \rightarrow [\kappa]_\theta^\mu$ is the assertion that, for every function $F : [\lambda]^\mu \rightarrow \theta$, there is $X \in [\lambda]^\kappa$ such that $F``[X]^\mu \neq \theta$. 

Remark

The question of whether $\lambda^+ \rightarrow [\lambda^+]_\omega^< \lambda^+$ (or even $\lambda^+ \rightarrow [\lambda^+]^2_\omega \lambda^+$) can hold if $\lambda$ is singular is a major open problem.
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The question of whether \( \lambda^+ \rightarrow [\lambda^+]^{<\omega}_{\lambda^+} \) (or even \( \lambda^+ \rightarrow [\lambda^+]^{2}_{\lambda^+} \)) can hold if \( \lambda \) is singular is a major open problem.
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**Theorem** (Tryba, Woodin)

If $\kappa$ is regular and $\kappa \rightarrow [\kappa]^{<\omega}_\kappa$, $\text{Refl}(\kappa)$ holds.

**Theorem** (Todorcevic)

If $\kappa$ is regular and $\kappa \rightarrow [\kappa]^2_\kappa$, then $\text{Refl}(\kappa)$ holds.

**Theorem** (Eisworth)

If $\lambda$ is singular and $\lambda^+ \rightarrow [\lambda^+]^{2\lambda+}_\lambda$, then every stationary subset of $\lambda^+$ reflects at arbitrarily high cofinalities.

**Question** (Eisworth)

Suppose $\lambda$ is a singular cardinal and $\text{Refl}(\lambda^+)$ holds. Must it be the case that every stationary subset of $\lambda^+$ reflects at arbitrarily high cofinalities?
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Proposition
Suppose \( \text{Refl}(\aleph_{\omega+1}) \) holds. Then every stationary subset of \( \aleph_{\omega+1} \) reflects at arbitrarily high cofinalities.
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Proof sketch
If $S \subseteq \aleph_{\omega+1}$, let $S' = \{\beta \mid S \text{ reflects at } \beta\}$. Note that, since every stationary set reflects, if $S$ is stationary, then $S'$ must also be stationary. Also note that if $S \subseteq S_{\aleph_n}^{\aleph_{\omega+1}}$, then $S' \subseteq S_{>\aleph_n}^{\aleph_{\omega+1}}$ and that, if $S'$ reflects at $\gamma$, then $S$ also reflects at $\gamma$. 
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Suppose \( \text{Refl}(\aleph_{\omega+1}) \) holds. Then every stationary subset of \( \aleph_{\omega+1} \) reflects at arbitrarily high cofinalities.

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If \( S \subseteq \aleph_{\omega+1} \), let \( S' = \{ \beta \mid S \text{ reflects at } \beta \} \). Note that, since every stationary set reflects, if \( S \) is stationary, then \( S' \) must also be stationary. Also note that if \( S \subseteq S_{\leq \aleph_n}^{\aleph_{\omega+1}} \), then \( S' \subseteq S_{> \aleph_n}^{\aleph_{\omega+1}} \) and that, if \( S' \) reflects at \( \gamma \), then \( S \) also reflects at \( \gamma \).

Now let \( S \subseteq \aleph_{\omega+1} \) be stationary, and let \( 0 < n < \omega \). To find \( \beta \in S_{\geq \aleph_n}^{\aleph_{\omega+1}} \) such that \( S \) reflects at \( \beta \), simply choose any \( \beta \in S^{(n)} \).
Approachability

Definition
Let $\mu$ be a singular cardinal. Suppose $2^\mu = \mu^+$, and let
$\vec{a} = \langle a_\alpha \mid \alpha < \mu^+ \rangle$ be an enumeration of the bounded subsets of $\mu^+$.
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1. A limit ordinal $\beta < \mu^+$ is approachable with respect to $\vec{a}$ if there is a cofinal $B \subseteq \beta$ such that $\text{otp}(B) = \text{cf}(\beta)$ and, for every $\alpha < \beta$, there is $\gamma < \beta$ such that $B \cap \alpha = a_\gamma$. 

Remarks
• If $\mu$ is a singular cardinal, then $\Box^{\ast} \mu \Rightarrow \text{AP}_\mu \Rightarrow$ all scales are good.
• If $n < \omega$, $\aleph_\omega \cdot m$ is strong limit for every $m \leq n$, $\text{Refl}(\aleph_\omega \cdot n + 1)$ holds, then $\text{AP}_{\aleph_\omega \cdot n}$ holds. This is not true of $\aleph_\omega ^2$. 

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2. The \textit{approachability property} holds at $\mu$ ($AP_\mu$) if the set of ordinals approachable with respect to $\bar{a}$ contains a club in $\mu^+$.

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• If $\mu$ is a singular cardinal, then $\Box^* \mu \Rightarrow AP_\mu \Rightarrow$ all scales are good.
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- If $\mu$ is a singular cardinal, then $\square^*_\mu \Rightarrow AP_\mu \Rightarrow$ all scales are good.
- If $n < \omega$, $\aleph_{\omega \cdot m}$ is strong limit for every $m \leq n$, $\text{Refl}(\aleph_{\omega \cdot n+1})$ holds, then $AP_{\aleph_{\omega \cdot n}}$ holds. This is not true of $\aleph_{\omega^2}$.
Theorem (Cummings, L-H)
Suppose there is an increasing sequence $\langle \kappa_i \mid i < \omega \cdot 2 \rangle$ of supercompact cardinals. Then there is a forcing extension in which $\text{Refl}(\aleph_{\omega \cdot 2+1})$ holds, but there is a stationary $S \subseteq S_{\aleph_{\omega \cdot 2+1}}$ that does not reflect at any ordinal in $S_{\geq \aleph_{\omega+1}}$. 
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Suppose there is an increasing sequence \( \langle \kappa_i \mid i < \omega \cdot 2 \rangle \) of supercompact cardinals. Then there is a forcing extension in which \( \text{Refl} (\aleph_{\omega \cdot 2 + 1}) \) holds, but there is a stationary \( S \subseteq S_{\aleph_{\omega \cdot 2 + 1}} \aleph_0 \) that does not reflect at any ordinal in \( S_{\aleph_{\omega \cdot 2 + 1}} \geq \aleph_{\omega + 1} \).

Proof Sketch
Assume GCH. Let \( \mu_0 = \sup (\{ \kappa_i \mid i < \omega \}) \), and let \( \mu_1 = \sup (\{ \kappa_i \mid i < \omega \cdot 2 \}) \). Let \( P_0 \) be the full-support iteration of length \( \omega \), \( \Coll (\omega, < \kappa_0) \ast \Coll (\kappa_0, < \kappa_1) \ast \Coll (\kappa_1, < \kappa_2) \ldots \) In \( V^{P_0} \), let \( P_1 \) be the full-support iteration of length \( \omega \), \( \Coll (\mu_0^+, < \kappa_\omega) \ast \Coll (\kappa_\omega, < \kappa_{\omega + 1}) \ldots \), and let \( P = P_0 \ast P_1 \).
Theorem (Cummings, L-H)
Suppose there is an increasing sequence $\langle \kappa_i \mid i < \omega \cdot 2 \rangle$ of supercompact cardinals. Then there is a forcing extension in which $\text{Refl}(\aleph_{\omega \cdot 2+1})$ holds, but there is a stationary $S \subseteq S_{\aleph_0}^{\aleph_{\omega \cdot 2+1}}$ that does not reflect at any ordinal in $S_{\aleph_{\omega+1}}^{\aleph_{\omega \cdot 2+1}}$.

Proof Sketch
Assume GCH. Let $\mu_0 = \sup(\{\kappa_i \mid i < \omega\})$, and let $\mu_1 = \sup(\{\kappa_i \mid i < \omega \cdot 2\})$. Let $P_0$ be the full-support iteration of length $\omega$, $\text{Coll}(\omega, < \kappa_0) \ast \text{Coll}(\kappa_0, < \kappa_1) \ast \text{Coll}(\kappa_1, < \kappa_2) \ldots$. In $V^{P_0}$, let $P_1$ be the full-support iteration of length $\omega$, $\text{Coll}(\mu_0^+, < \kappa_\omega) \ast \text{Coll}(\kappa_\omega, < \kappa_{\omega+1}) \ldots$, and let $P = P_0 \ast P_1$. In $V^P$, we have $\mu_0 = \aleph_\omega$, $(\mu_0^+)^V = \aleph_{\omega+1}$, $\mu_1 = \aleph_{\omega \cdot 2}$, $(\mu_1^+)^V = \aleph_{\omega \cdot 2+1}$. 
In $\mathcal{V}^\mathbb{P}$, let $\bar{a} = \langle a_\alpha \mid \alpha < \mu_1^+ \rangle$ be an enumeration of the bounded subsets of $\mu_1^+$. Let $\mathbb{Q}$ be the forcing poset whose conditions are closed, bounded subsets of $\mu_1^+$ all of whose members are approachable with respect to $\bar{a}$. $\mathbb{Q}$ is ordered by end-extension.
In $V^\mathbb{P}$, let $\vec{a} = \langle a_\alpha \mid \alpha < \mu_1^+ \rangle$ be an enumeration of the bounded subsets of $\mu_1^+$. Let $Q$ be the forcing poset whose conditions are closed, bounded subsets of $\mu_1^+$ all of whose members are approachable with respect to $\vec{a}$. $Q$ is ordered by end-extension.

**Facts**

1. (Shelah) $Q$ is strongly ($< \mu_1$)-strategically closed and forces $AP_{\mu_1}$. 
In $V^P$, let $\bar{a} = \langle a_\alpha \mid \alpha < \mu_1^+ \rangle$ be an enumeration of the bounded subsets of $\mu_1^+$. Let $Q$ be the forcing poset whose conditions are closed, bounded subsets of $\mu_1^+$ all of whose members are approachable with respect to $\bar{a}$. $Q$ is ordered by end-extension.

**Facts**

1. (Shelah) $Q$ is strongly $(< \mu_1)$-strategically closed and forces $AP_{\mu_1}$.

2. (Hayut) In $V^{P*Q}$, $\text{Refl}(\mu_1^+)$ holds.
In $V^{P*Q}$, let $\mathbb{S}$ be the forcing whose conditions are functions $s : \gamma \to 2$ such that:

1. $\gamma < \mu + 1$.
2. If $s(\alpha) = 1$, then $\text{cf}(\alpha) = \omega$.
3. For every $\beta \in \mathbb{S}$, $\mu + 1 \geq \mu + 0$, $\{ \alpha < \gamma \mid s(\alpha) = 1 \} \cap \beta$ is not stationary.

$\mathbb{S}$ is ordered by reverse inclusion. $\mathbb{S}$ is easily seen to preserve all cardinals and add a stationary subset of $\mathbb{S}$. The bulk of the proof, which will be omitted, lies in showing that it is still the case that $\text{Refl}(\mu + 1)$ holds after forcing with $\mathbb{S}$. 
In $V^{P*Q}$, let $S$ be the forcing whose conditions are functions $s : \gamma \to 2$ such that:

1. $\gamma < \mu_1^+$. 
2. If $s(\alpha) = 1$, then $\text{cf}(\alpha) = \omega$. 
3. For every $\beta \in S_{\geq \mu_0^+}^{\mu_1^+}$, \{\(\alpha \prec \gamma \mid s(\alpha) = 1\}\} \cap \beta$ is not stationary.

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In $V^{P\ast Q}$, let $S$ be the forcing whose conditions are functions $s : \gamma \to 2$ such that:

1. $\gamma < \mu_1^+$. 
2. If $s(\alpha) = 1$, then $\text{cf}(\alpha) = \omega$. 
3. For every $\beta \in S_{\geq \mu_0^+}^{\mu_1^+}$, \(\{\alpha < \gamma \mid s(\alpha) = 1\}\cap \beta\) is not stationary.

$S$ is ordered by reverse inclusion. $S$ is easily seen to preserve all cardinals and add a stationary subset of $S_{\omega_1^+}^{\mu_1^+}$ that does not reflect at any ordinals in $S_{\geq \mu_0^+}^{\mu_1^+}$. The bulk of the proof, which will be omitted, lies in showing that it is still the case that $\text{Refl}(\mu_1^+)$ holds after forcing with $S$. \(\square\)
Some variations

**Theorem (L-H)**

Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$, we have the following:

1. $\text{Refl}(\mu^+)$. 

2. There is a stationary subset $S \subseteq S^\mu_\omega$ that does not reflect at any ordinals in $S^\mu_\omega \geq \aleph_{\omega+1}$. 

**Theorem (L-H)**

Suppose there is an $\omega \cdot 2$-sequence of supercompact cardinals. Then there is a forcing extension in which:

1. $\text{Refl}(\aleph_{\omega \cdot 2}^+)$. 

2. For every stationary $S \subseteq S^{\aleph_{\omega \cdot 2}}_\omega$, there is a stationary $T \subseteq S^{\aleph_{\omega \cdot 2}}_\omega$ such that $T$ does not reflect at any ordinals in $S^{\aleph_{\omega \cdot 2}}_\omega \geq \aleph_{\omega+1}$. 

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**Theorem** (L-H)
Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$, we have the following:
1. $\text{Refl}(\mu^+)$.  
2. There is a stationary subset $S \subseteq S_\mu^+$ that does not reflect at any ordinals in $S_{\geq \aleph_{\omega+1}}^\mu$.  

**Theorem** (L-H)
Suppose there is a $\omega \cdot 2$-sequence of supercompact cardinals. Then there is a forcing extension in which:
1. $\text{Refl}(\aleph_{\omega \cdot 2 + 1})$.  
2. For every stationary $S \subseteq S_{\aleph_{\omega \cdot 2 + 1}}^{\aleph_\omega}$, there is a stationary $T \subseteq S$ such that $T$ does not reflect at any ordinals in $S_{\geq \aleph_{\omega+1}}^{\aleph_{\omega \cdot 2 + 1}}$.  

Results without approachability

**Theorem** (L-H)

Suppose there is an $\omega \cdot 2$-sequence of supercompact cardinals, with $\mu_0$ the supremum of the first $\omega$ and $\mu_1$ the supremum of the entire sequence. Then there is a cardinal-preserving forcing extension in which:

1. Refl($\mu_1^+$).
2. There is a stationary subset of $S^{\mu_1^+}_\omega$ that does not reflect at any ordinals in $S^{\mu_1^+}_{\geq \mu_0^+}$.
3. $AP_{\mu_1}$ fails.
Results without approachability

**Theorem (L-H)**

Suppose there is an $\omega \cdot 2$-sequence of supercompact cardinals, with $\mu_0$ the supremum of the first $\omega$ and $\mu_1$ the supremum of the entire sequence. Then there is a cardinal-preserving forcing extension in which:

1. $\text{Refl}(\mu_1^+)$.  
2. There is a stationary subset of $S_{\omega_1}^{\mu_1^+}$ that does not reflect at any ordinals in $S_{\geq \mu_0^+}^{\mu_1^+}$.  
3. $AP_{\mu_1}$ fails.

**Theorem (L-H)**

Under the same hypotheses, there is a forcing extension in which (1),(2), and (3) hold as above, $\mu_0 = \aleph_{\omega^2}$, and $\mu_1 = \aleph_{\omega^2 + 2}$. 
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Is it consistent that $\text{Refl}(\aleph_{\omega^2 + 1})$ holds and, for every stationary $S \subseteq \aleph_{\omega^2 + 1}$, there is a stationary $T \subseteq S$ that does not reflect at arbitrarily high cofinalities?

• Is it consistent that $\text{Refl}(\aleph_{\omega \cdot 2 + 1})$ holds and there is a stationary subset of $\aleph_{\omega \cdot 2 + 1}$ that reflects only at ordinals of cofinality $\aleph_n$ for $n$ even?

• Is it consistent that $\text{Refl}(\aleph_{\omega^1 \cdot 2 + 1})$ holds and there is a stationary subset of $\aleph_{\omega^1 \cdot 2 + 1}$ that only reflects at ordinals in $\aleph_{\omega^1 \cdot 2 + 1} \geq \aleph_{\omega + 1}$?
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What about other patterns of reflection? For example:
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**Question**
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Is it consistent that \( \text{Refl}(\aleph_{\omega^2+1}) \) holds and, for every stationary \( S \subseteq \aleph_{\omega^2+1} \), there is a stationary \( T \subseteq S \) that does not reflect at arbitrarily high cofinalities?

**Question**
What about other patterns of reflection? For example:

- Is it consistent that \( \text{Refl}(\aleph_{\omega+1}) \) holds and there is a stationary subset of \( \aleph_{\omega+1} \) that reflects only at ordinals of cofinality \( \aleph_n \) for \( n \) even?
Questions

**Question**
Is it possible to bring the result of the previous theorem down to $\aleph_{\omega^2+1}$?

**Question**
Is it consistent that $\text{Refl}(\aleph_{\omega^2+1})$ holds and, for every stationary $S \subseteq \aleph_{\omega^2+1}$, there is a stationary $T \subseteq S$ that does not reflect at arbitrarily high cofinalities?

**Question**
What about other patterns of reflection? For example:

- Is it consistent that $\text{Refl}(\aleph_{\omega+1})$ holds and there is a stationary subset of $\aleph_{\omega+1}$ that reflects only at ordinals of cofinality $\aleph_n$ for $n$ even?

- Is it consistent that $\text{Refl}(\aleph_{\omega \cdot 2+1})$ holds and there is a stationary subset of $S^\aleph_{\omega \cdot 2+1}$ that only reflects at ordinals in $S^\aleph_{\omega^2+1}$?
Thank you