Patterns of stationary reflection

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If $\kappa < \lambda$ are infinite cardinals, with κ regular, then $S^\lambda_\kappa = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$.

Remark

If $S \subseteq S^\lambda_\kappa$ and S reflects at β, then $\mathrm{cf}(\beta) > \kappa$. Thus, if κ is regular and $S \subseteq S^{\kappa^+}_\kappa$, then S does not reflect.

Classical results

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If μ is a singular limit of supercompact cardinals, then $\mathrm{Refl}(\mu^+)$ holds.

Theorem (Magidor)
If $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, then there is a forcing extension in which $\kappa_n = \aleph_{n+1}$ for every $n < \omega$ and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Square-bracket partition relations

Definition

1. If λ is an infinite, regular cardinal and $S \subseteq \lambda$ is stationary, we say S *reflects at arbitrarily high cofinalities* if, for every regular $\kappa < \lambda$, there is $\beta \in S^{\lambda}_{\geq\kappa}$ such that S reflects at β.

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3. $\lambda \to [\kappa]^\mu_\theta$ is the assertion that, for every function $F : [\lambda]^\mu \to \theta$, there is $X \in [\lambda]^\kappa$ such that $F``[X]^\mu \neq \theta$.

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Remark

The question of whether $\lambda^+ \to [\lambda^+]^{<\omega}_{\lambda^+}$ (or even $\lambda^+ \to [\lambda^+]^{2}_{\lambda^+}$) can hold if λ is singular is a major open problem.

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Theorem (Eisworth)
If λ is singular and $\lambda^+ \to [\lambda^+]^2_{\lambda^+}$, then every stationary subset of λ^+ reflects at arbitrarily high cofinalities.

Question (Eisworth)
Suppose λ is a singular cardinal and $\mathrm{Refl}(\lambda^+)$ holds. Must it be the case that every stationary subset of λ^+ reflects at arbitrarily high cofinalities?

$\aleph_{\omega+1}$

Proposition

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Proof sketch

If $S \subseteq \aleph_{\omega+1}$, let $S' = \{\beta \mid S \text{ reflects at } \beta\}$. Note that, since every stationary set reflects, if S is stationary, then S' must also be stationary. Also note that if $S \subseteq S^{\aleph_{\omega+1}}_{\aleph_n}$, then $S' \subseteq S^{\aleph_{\omega+1}}_{>\aleph_n}$ and that, if S' reflects at γ, then S also reflects at γ.

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Now let $S \subseteq \aleph_{\omega+1}$ be stationary, and let $0 < n < \omega$. To find $\beta \in S^{\aleph_{\omega+1}}_{\geq\aleph_n}$ such that S reflects at β, simply choose any $\beta \in S^{(n)}$.

Approachability

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1. A limit ordinal $\beta < \mu^+$ is *approachable with respect to* $\vec{a}$ if there is a cofinal $B \subseteq \beta$ such that $\mathrm{otp}(B) = \mathrm{cf}(\beta)$ and, for every $\alpha < \beta$, there is $\gamma < \beta$ such that $B \cap \alpha = a_\gamma$.

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Remarks

- If μ is a singular cardinal, then $\Box^*_\mu \Rightarrow AP_\mu \Rightarrow$ all scales are good.
- If $n < \omega$, $\aleph_{\omega\cdot m}$ is strong limit for every $m \leq n$, $\mathrm{Refl}(\aleph_{\omega\cdot n+1})$ holds, then $AP_{\aleph_{\omega\cdot n}}$ holds. This is not true of $\aleph_{\omega^2}$.

$\aleph_{\omega\cdot2+1}$

Theorem (Cummings, L-H)

Suppose there is an increasing sequence $\langle \kappa_i \mid i < \omega \cdot 2 \rangle$ of supercompact cardinals. Then there is a forcing extension in which $\mathrm{Refl}(\aleph_{\omega\cdot2+1})$ holds, but there is a stationary $S \subseteq S^{\aleph_{\omega\cdot2+1}}_{\aleph_0}$ that does not reflect at any ordinal in $S^{\aleph_{\omega\cdot2+1}}_{\geq\aleph_{\omega+1}}$.

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Proof Sketch

Assume GCH. Let $\mu_0 = \sup(\{\kappa_i \mid i < \omega\})$, and let $\mu_1 = \sup(\{\kappa_i \mid i < \omega \cdot 2\})$. Let $\mathbb{P}_0$ be the full-support iteration of length ω, $\mathrm{Coll}(\omega, < \kappa_0) * \mathrm{Coll}(\kappa_0, < \kappa_1) * \mathrm{Coll}(\kappa_1, < \kappa_2) \ldots$ In $V^{\mathbb{P}_0}$, let $\mathbb{P}_1$ be the full-support iteration of length ω, $\mathrm{Coll}(\mu_0^+, < \kappa_\omega) * \mathrm{Coll}(\kappa_\omega, < \kappa_{\omega+1}) \ldots$, and let $\mathbb{P} = \mathbb{P}_0 * \mathbb{P}_1$.

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In $V^{\mathbb{P}}$, let $\vec{a} = \langle a_\alpha \mid \alpha < \mu_1^+ \rangle$ be an enumeration of the bounded subsets of μ_1^+. Let $\mathbb{Q}$ be the forcing poset whose conditions are closed, bounded subsets of μ_1^+ all of whose members are approachable with respect to $\vec{a}$. $\mathbb{Q}$ is ordered by end-extension.

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Facts

1. (Shelah) $\mathbb{Q}$ is strongly $(< \mu_1)$-strategically closed and forces AP_{μ_1}.
2. (Hayut) In $V^{\mathbb{P} * \mathbb{Q}}$, $\mathrm{Refl}(\mu_1^+)$ holds.

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1. $\gamma < \mu_1^+$.
2. If $s(\alpha) = 1$, then $\mathrm{cf}(\alpha) = \omega$.
3. For every $\beta \in S^{\mu_1^+}_{\geq \mu_0^+}$, $\{\alpha < \gamma \mid s(\alpha) = 1\} \cap \beta$ is not stationary.

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$\mathbb{S}$ is ordered by reverse inclusion.

$\mathbb{S}$ is easily seen to preserve all cardinals and add a stationary subset of $S^{\mu_1^+}_{\omega}$ that does not reflect at any ordinals in $S^{\mu_1^+}_{\geq \mu_0^+}$. The bulk of the proof, which will be omitted, lies in showing that it is still the case that $\mathrm{Refl}(\mu_1^+)$ holds after forcing with $\mathbb{S}$. □

Theorem (L-H)

Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$, we have the following:

1. $\mathrm{Refl}(\mu^+)$.
2. There is a stationary subset $S \subseteq S^{\mu^+}_\omega$ that does not reflect at any ordinals in $S^{\mu^+}_{\geq \aleph_{\omega+1}}$.

Some variations

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Theorem (L-H)

Suppose there is an $\omega \cdot 2$-sequence of supercompact cardinals. Then there is a forcing extension in which:

1. $\mathrm{Refl}(\aleph_{\omega \cdot 2+1})$.
2. For every stationary $S \subseteq S^{\aleph_{\omega \cdot 2+1}}_{<\aleph_\omega}$, there is a stationary $T \subseteq S$ such that T does not reflect at any ordinals in $S^{\aleph_{\omega \cdot 2+1}}_{\geq \aleph_{\omega+1}}$.

Results without approachability

Theorem (L-H)

Suppose there is an $\omega \cdot 2$-sequence of supercompact cardinals, with μ_0 the supremum of the first ω and μ_1 the supremum of the entire sequence. Then there is a cardinal-preserving forcing extension in which:

1. $\mathrm{Refl}(\mu_1^+)$.
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3. AP_{μ_1} fails.

Theorem (L-H)

Under the same hypotheses, there is a forcing extension in which (1),(2), and (3) hold as above, $\mu_0 = \aleph_{\omega^2}$, and $\mu_1 = \aleph_{\omega^2 \cdot 2}$.

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What about other patterns of reflection? For example:

- Is it consistent that $\mathrm{Refl}(\aleph_{\omega+1})$ holds and there is a stationary subset of $\aleph_{\omega+1}$ that reflects only at ordinals of cofinality $\aleph_n$ for n even?

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- Is it consistent that $\mathrm{Refl}(\aleph_{\omega\cdot2+1})$ holds and there is a stationary subset of $S^{\aleph_{\omega\cdot2+1}}_{\omega}$ that only reflects at ordinals in $S^{\aleph_{\omega\cdot2+1}}_{\geq\aleph_{\omega+1}}$?

Thank you