Splitting, bounding and almost disjointness number

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Some classical cardinal invariants of the continuum

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- The *dominating number* $\mathfrak{d}$ is the least size of a $\leq^*$-cofinal family.

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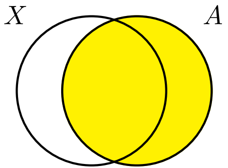

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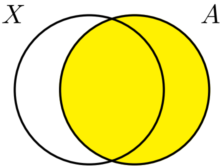

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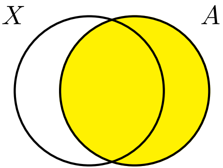

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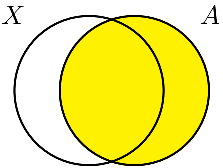

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- $\mathcal{A} \subseteq [\omega]^\omega$ is an *a.d. (almost disjoint) family* if any two distinct members of $\mathcal{A}$ have finite intersection.
- the *almost disjointness number* $\mathfrak{a}$ is the minimal size of an infinite mad (maximal a.d.) family.

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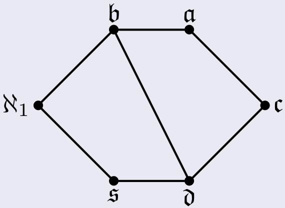

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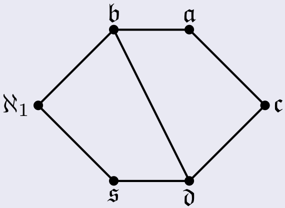

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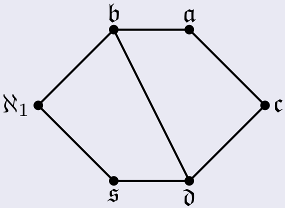

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Additionally, if $\theta < \mu$ is uncountable regular, if we alternate $\mathbb{D}$ with Mathias-Prikry posets of size $< \theta$ (i.e., $\mathbb{M}_F$ with F a filter base of size $< \theta$) by a book-keeping devise, the resulting iteration forces $\mathfrak{s} = \theta < \mathfrak{b} = \mu$.

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For $x \in L$, $\hat{\mathcal{I}}_x$ denotes the ideal (on L_x) generated by $\mathcal{I}_x$.

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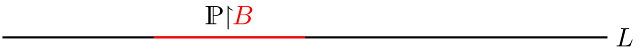

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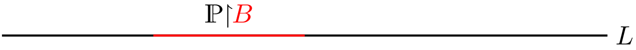

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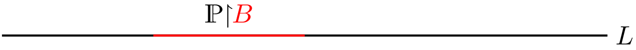

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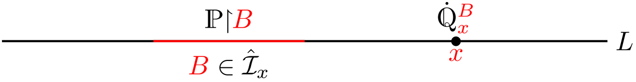

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(ii) If A has a maximum x but $A_x \notin \hat{\mathcal{I}}_x$, then $\mathbb{P}{\restriction}A = \mathrm{limdir}\{\mathbb{P}{\restriction}B \ / \ B \subseteq A \text{ and } B \cap L_x \in \mathcal{I}_x{\restriction}A\}$.

Iterations along a template

Iterations we are interested in: Fix θ an uncountable regular cardinal, an indexed template $\langle L, \bar{\mathcal{I}} \rangle$, H, M disjoint sets, $L = H \cup M$ and $C_x \in \hat{\mathcal{I}}_x$ of size $< \theta$ for $x \in M$. For $A \subseteq L$, by recursion on $\mathrm{Dp}(A)$, define a poset $\mathbb{P}{\restriction}A$ as follows:

(i) If A has a maximum x and $A_x = A \cap L_x \in \hat{\mathcal{I}}_x$ then $\mathbb{P}{\restriction}A = \mathbb{P}{\restriction}A_x * \dot{\mathbb{Q}}_x^{A_x}$ where:

- If $x \in H$ then $\dot{\mathbb{Q}}_x^{A_x} = \mathbb{D}^{V^{\mathbb{P}{\restriction}A_x}}$.
- If $x \in M$ and $C_x \subseteq A_x$, fix $\dot{F}_x$ is a $\mathbb{P}{\restriction}C_x$-name of a filter base of size $< \theta$ (this only depends on C_x) and put $\dot{\mathbb{Q}}_x^{A_x} = \mathbb{M}_{\dot{F}_x}$.
- If $x \in M$ but $C_x \nsubseteq A_x$, then $\dot{\mathbb{Q}}_x^{A_x} = \mathbb{1}$.

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(iii) If A does not have a maximum, then $\mathbb{P}{\restriction}A = \mathrm{limdir}\{\mathbb{P}{\restriction}B \ / \ \exists_{x \in A}(B \in \mathcal{I}{\restriction}A)\}$ (so $\mathbb{P}{\restriction}\varnothing = \mathbb{1}$).

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(b) *if $p \in \mathbb{P}\restriction A$ and $\dot{x}$ is a $\mathbb{P}\restriction A$-name for a real, then there is $C \subseteq A$ of size $< \theta$ such that $p \in \mathbb{P}\restriction C$ and $\dot{x}$ is a $\mathbb{P}\restriction C$-name.*

A consistency result

Theorem (M.)

Let $\theta < \kappa < \mu < \lambda$ be uncountable regular cardinals where κ is measurable, $\theta^{<\theta} = \theta$ and $\lambda^\kappa = \lambda$. Then, there exists a ccc poset forcing $\mathfrak{s} = \theta < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

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Problem

Can a similar consistency result be proven with ZFC alone?

Shelah's template

Fix uncountable regular cardinals $\theta < \mu < \lambda$. For $\delta \leq \lambda$ define

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linearly ordered by $x < y$ iff one of the following holds:

(i) there is some $k < \min\{|x|, |y|\}$ such that $x \restriction k = y \restriction k$ and $x(k) < y(k)$;

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The family $\mathcal{I}^\delta$ if formed by finite unions of sets from

$$\{L^\delta_\alpha \;/\; \alpha \in \lambda\mu\} \cup \{[x{\restriction}(|x|-1), x) \;/\; x \in L^\delta \text{ is } \theta\text{-relevant}\} \cup \{\{z\} \;/\; z \in L^\delta\}.$$

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$\langle L^\delta, \bar{\mathcal{I}}^\delta \rangle$ is an indexed template, where $\mathcal{I}^\delta_x := \{A \in \mathcal{I}^\delta \;/\; A \subseteq L^\delta_x\}$.

Isomorphism-of-name arguments

Shelah proved that, assuming CH and $\lambda^{\aleph_0} = \lambda$ (regular), an iteration of $\mathbb{D}$ along L^λ (with $\theta = \aleph_1$) forces $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

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2. By a Δ-system argument, wlog assume that $\{B_\alpha \ / \ \alpha < \omega_1\}$ forms a Δ-system with root R, $\langle B_\alpha, \bar{\mathcal{I}}^\lambda{\restriction}B_\alpha\rangle \cong \langle B_0, \bar{\mathcal{I}}^\lambda{\restriction}B_0\rangle$ and $\langle \mathbb{P}{\restriction}B_\alpha, \dot{a}_\alpha\rangle \cong \langle \mathbb{P}{\restriction}B_0, \dot{a}_0\rangle$ for all $\alpha < \omega_1$.

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3' Find $\delta' \in (\delta, \lambda)$, choose a suitable $B_\kappa \subseteq L^{\delta'}$ such that $B_\kappa \cap L^\delta = R$ (the same intersected with all B_α with $\alpha < \theta$) and extend the iteration $\mathbb{P}{\restriction}L^\delta$ to $\mathbb{P}{\restriction}L^{\delta'}$ such that $\langle \mathbb{P}{\restriction}B_0, \dot{a}_0 \rangle \simeq \langle \mathbb{P}{\restriction}B_\kappa, \dot{a}_\kappa \rangle$.

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Let $\theta^+ < \delta < \lambda$, $\mathbb{P}^\delta$ be an iteration of $\mathbb{D}$ and Mathias-Prikry forcings of size $< \theta$ along L^δ and $\dot{A}$ a $\mathbb{P}{\restriction}L^\delta$-name of an a.d. family of size $\kappa \in (\theta, \lambda)$.

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(c) $\mathbb{P}^{\delta'}{\restriction}L^{\delta'}$ forces that $\dot{A}$ is not mad.

Main result

Theorem (Fischer and M.)

There is an iteration $\mathbb{P}^\lambda$ *along* L^λ *that forces*
$\mathfrak{s} = \theta < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda.$

Further results

Theorem (Fischer and M.)

If $\theta_0 < \theta_1 < \theta < \mu < \lambda$ *are uncountable regular,* $\theta^{<\theta} = \theta$ *and* $\lambda^{<\lambda} = \lambda$*, then there is a ccc poset that forces* $\mathrm{add}(\mathcal{N}) = \theta_0 < \mathrm{cov}(\mathcal{N}) = \theta_1 < \mathfrak{p} = \mathfrak{g} = \mathfrak{s} = \theta < \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu < \mathrm{non}(\mathcal{N}) = \mathfrak{a} = \mathfrak{r} = \mathfrak{c} = \lambda$.