Avoiding rational distances

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Komjáth's question

Let $X \subseteq \mathbb{R}^n$. Must there exist a subset Y of X such that X and Y have same Lebesgue outer measure and the distance between any two points of Y is irrational?

Some remarks

- If X is Borel, such a Y can always be found - List all positive measure compact subsets of X and inductively choose a point from each one of them which is not at a rational distance from the previously chosen points.

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- Komjath showed that one can color $\mathbb{R}^n$ by countably many colors such that no two points of same color are at a rational distance. Hence there is always a positive outer measure subset avoiding rational distances.

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- Komjath showed that one can color $\mathbb{R}^n$ by countably many colors such that no two points of same color are at a rational distance. Hence there is always a positive outer measure subset avoiding rational distances.
- We showed that when $n = 1$ the answer is yes.
- We don't know the answer in higher dimensions.

Sometimes ZFC is not enough

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- Komjáth constructed a model in which there is a non meager subset X of plane such that every non meager subset Y of X contains three collinear points and the vertices of a right triangle.
- Shelah has obtained similar results for measure.

Forcing

The main tool in the proof is a result of Gitik and Shelah on forcing with sigma ideals which says the following:

Theorem

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Corollary

Suppose T is a subtree of $\omega^{<\omega}$ such that every node in T has at least two children and $X \subseteq \mathbb{R}^n$. Suppose $\langle X_\sigma : \sigma \in T \rangle$ satisfies

- $X_{\langle\rangle} = X$
- *For each $\sigma \in T$, $X_\sigma = \bigsqcup \{X_{\sigma n} : n < \omega, \sigma n \in T\}$ and*
- *X_σ has full outer measure in X*

Then there exists $Y \subseteq X$ such that Y has full outer measure in X and for each $\sigma \in T$, $X_\sigma \setminus Y$ has full outer measure in X.

References

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