Open-open game of uncountable lenght

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The following game was introduced by

P. Daniels, K. Kunen and H. Zhou

On the open-open game, Fund. Math. 145 (1994), no. 3, 205 - 220.

Two players playing on a topological space X.

- Player I choosing a non-empty open set $U \subseteq X$.
- Player II should choosing a non-empty open set $V \subseteq U$
- Player I wins if the union of all open sets which have been chosen by Player II is dense in X

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Let X be a topologigal space equipped with a topology $\mathcal{T}$.

The space X is called κ-I-*favorable* whenever there exists a function

$$\sigma : \bigcup\{\mathcal{T}^\alpha : \alpha \in \kappa\} \to \mathcal{T}$$

such that for each sequence $\{B_\alpha : \alpha \in \kappa\}$ consisting of elements of $\mathcal{T}$ with $B_0 \subseteq \sigma(\emptyset)$ and

$$B_\alpha \subseteq \sigma(\{B_\beta : \beta < \alpha\}) \text{ for each } \alpha \in \kappa,$$

the union $\bigcup\{B_\alpha : \alpha \in \kappa\}$ is dense in X.

The function σ is called a $\kappa-$*winning strategy*.

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Let X be topological space. Then

$$\mu(X) = \min\{\kappa \in \text{Card} : X \text{ is } \kappa\text{-I-favorable}\}.$$

and

$$\text{sat}(X) = \min\{\tau : |\mathcal{B}| < \tau \text{ for each family } \mathcal{B} \text{ of pairwise disjoint open set}\}$$

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The next theorem is analog of theorem 1 from A. Błaszczyk, *Souslin number and inverse limits*, Topology and measure III, Proc. Conf., Vitte/Hiddensee 1980, Part 1, 21 – 26 (1982).

A directed set Σ is said to be κ-complete if any chain of length κ consisting of its elements has least upper bound in Σ.

An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a κ-complete, whenever Σ is κ-complete and for every chain $\{\sigma_\alpha : \alpha \in \kappa\} \subseteq \Sigma$, such that $\sigma = \sup\{\sigma_\alpha : \alpha \in \kappa\} \in \Sigma$, there holds

$$X_\sigma = \varprojlim\{X_{\sigma_\alpha}, \pi^{\sigma_\alpha}_{\sigma_\beta}\}.$$

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A continuous surjection is called *skeletal* whenever for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior.

Theorem

Let X be topological space and $\kappa = \mu(X)$. If X can be represented as an inverse limit of κ-complete system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ and all bounding map are skeletal surjection then $\mu(X) = \sup\{\mu(X_\sigma) : \sigma \in \Sigma\}$.

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For each cardinal number κ consider a class $\mathcal{C}_\kappa$ of topological spaces such that each of them can be represented as an inverse limit of κ-complete system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ with $\mathrm{w}(X_\sigma) \leqslant \kappa$ and each X_σ is T_0 space and bounding maps π^σ_ϱ are skeletal surjection.

Theorem

The class $\mathcal{C}_\kappa$ is closed under any product.

Theorem

If X belongs to the class $\mathcal{C}_\kappa$ then $\mathrm{c}(X) \leqslant \kappa$ and $\mu(X) \leqslant \kappa$.

Theorem

Each Tichonov space X can be dense embeded into topological space which belong to the class $\mathcal{C}_\kappa$ where $\kappa = \mu(X)^{<\mu(X)}$ and each X_σ is Tichonov.

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The space X is called I-*favorable* whenever it is ω-I-favorable

Theorem (Sz. Plewik and me 2008)

Let X be compact space.X is a I-favorable , iff

$$X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\},$$

where $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, all spaces X_σ are compact and metrizable, and all bonding maps π^σ_ϱ are skeletal.

Proposition(Sz. Plewik and me 2008)

If X is a I-favorable completely regular space then X can be dense embeding into $Y = \varprojlim\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a σ-complete inverse system, all spaces Y_σ are metrizable, and all bonding maps π^σ_ϱ are skeletal.

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Proposition

Let X be topological space then

$$c(X) \leqslant \mu(X) \leqslant \mathrm{sat}(X) \leqslant c(X)^+$$

W.G. Fleissner, *Some spaces related to topological inequalities proven by the Erdös-Rado theorem*, Proc. Amer. Math. Soc., 71 (1978), 313–320

W.G. Fleissner has constructed the space W such that $c(W) = \aleph_0$ and $c(W \times W \times W) = \aleph_2$. By previous proposition we could not request that each topological space X with $\mu(X) = \kappa$ can be dense embeding into $Y = \varprojlim\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\{Y_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a κ-complete inverse system, all spaces Y_σ have weight equal κ, and all bonding maps π^σ_ϱ are skeletal.

Example

Each linear ordered topological space X(LOTS) can be dense embeded into topological space which belongs to the class $\mathcal{C}_\kappa$ where $\kappa = \mathrm{d}(X)$