A class of invisible spaces

Quidquid latine dictum sit, altum videtur

K. P. Hart

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Hejnice, 1. Unor, 2015: 10:00 – 10:45

Small diagonals

Definition (Van Douwen, Hušek, Zhou)

A space, X, has a *small diagonal* if every uncountable subset of $X^2 \setminus \Delta(X)$ has an uncountable subset whose closure is disjoint from $\Delta(X)$.

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Hušek defined the negation: X has an ω_1-accessible diagonal is there if a sequence $\langle \langle x_\alpha, y_\alpha \rangle : \alpha \in \omega_1 \rangle$ that converges to $\Delta(X)$

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A sufficient condition

If $\Delta(X)$ is a G_δ-set then X has a small diagonal.

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If $\Delta(X)$ is a G_δ-set then X has a small diagonal.
Hence, for example, metrizable spaces have small diagonals.

Raison d'etre

Theorem (Hušek, special case)

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Assume not . . . Contradiction.

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Take a sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary substructures of a suitable $H(\theta)$ such that

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Apply "Assume not" at each α: there are x_α and y_α that have the same coordinates in $I \cap M_\alpha$ but satisfy $f(x_\alpha) \neq f(y_\alpha)$.

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Details (continued)

Take basic neighbourhoods V_α and W_α of x_α and y_α such that $f[V_\alpha] \cap f[W_\alpha] = \emptyset$.

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Then still $f(x_\alpha) \neq f(y_\alpha)$ but now the points differ in finitely many places.

By elementarity x_α, y_α, V_α and W_α may be taken in $M_{\alpha+1}$, and so $F_\alpha \subseteq M_{\alpha+1} \setminus M_\alpha$.

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Contradiction!

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Many proofs of results on csD spaces (compact small Diagonal) work like this.

Elementary sequences

Many proofs of results on csD spaces (compact small Diagonal) work like this.

Let X be compact; a sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable elementary substructures, as above, with $X \in M_0$ is an *elementary sequence for* X.

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Elementary sequences of pairs

We assume X to be embedded into $[0,1]^\kappa$, say $\kappa = w(X)$, so the following makes sense.

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One of the coordinate sequences may be constant.

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Now go back to the proof and recognize all these ingredients.

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We are lead to ask

Is every csD space metrizable?

I will discuss various positive consistency results.

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There is as yet no consistent *negative* answer.

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small Diagonals and metrizability

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There is as yet no consistent *negative* answer.

This explains the title of this talk: there are no illuminating examples of csD spaces; for all we know they are all metrizable.

Positive answers

If X is csD and $w(X) \leqslant \aleph_1$ then X is metrizable

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Then $x_\alpha \in V$ iff $y_\alpha \in V$ for $\alpha \geqslant \delta$, hence $x \in \operatorname{cl}\{x_\alpha : \alpha \in A\} \cap \operatorname{cl}\{y_\alpha : \alpha \in A\}$.

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Continuum Hypothesis

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Countable tightness: it implies separability in this case

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Hence: csD spaces have countable tightness

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Then $\bigl\langle \{x, x_\alpha\} : \alpha \in \omega_1 \bigr\rangle$ would not be ω_1-separated.

Hence: csD spaces have countable tightness and the Continuum Hypothesis implies csD spaces are metrizable.

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Proper Forcing Axiom

Dow and Pavlov: PFA implies csD spaces are metrizable.

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I'm not even attempting to sketch the proof.

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Take an elementary sequence of pairs and $A \subseteq \omega_1$ uncountable. Then $\{x_\alpha : \alpha \in A\}$ is Lindelöf.

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$M_{\delta+1}$ contains a countable local base, $\mathcal{B}$, at x_δ.
For each member B of $\mathcal{B}$ and $\alpha > \delta$ we have $x_\alpha \in B$ iff $y_\alpha \in B$.

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We do need a countable local base at x to make the last part work.

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Hence we can choose $x_\alpha \in M_{\alpha+1}$ such that $x_\alpha \neq x$, but $x_\alpha \restriction M_\alpha = x \restriction M_\alpha$.

Let $A \subseteq \omega_1$ be uncountable and let $y \in M$ be a complete accumulation point of $\{x_\alpha : \alpha \in A\}$.

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By elementarity: $x = y$.

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Contradiction.

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In particular csD spaces are ω_1-free.

A consistent “Yes” to Hušek

In any extension of a model of the Continuum Hypothesis by a property K forcing every ω_1-free compact space is L-reflecting (there is an elementary sequence for it such that $X \cap M$ is Lindelöf) and (hence) first-countable.

A consistent "Yes" to Hušek

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In particular csD spaces are metrizable in these extensions.

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However

The main question remains: are compact csD spaces metrizable.

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(Many partial questions do not have answers yet either: somewhere/everywhere first-countable, what is the weight, ...)

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However

The main question remains: are compact csD spaces metrizable.

(Many partial questions do not have answers yet either: somewhere/everywhere first-countable, what is the weight, . . .)

I will be expecting solutions from you next year.

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Light reading

Website: `fa.its.tudelft.nl/~hart`

Alan Dow and Klaas Pieter Hart,
Elementary chains and compact spaces with a small diagonal, Indagationes Mathematicae, **23** (2012), 438–447.

Alan Dow and Klaas Pieter Hart,
Reflecting Lindelöf and converging ω_1-sequences, Fundamenta Mathematicae, **224** (2014) 205–218.

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