A Survey of the Density Topology

Stuart Nygard

Boise State University

Advisor: Dr. Andrés Caicedo

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Outline

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Motivation

Suppose we have a function, continuous when restricted to a set of full measure. The continuous functions are strictly contained in this collection.

For example, consider the functions

$$\chi_{\mathbb{Q}}(x), \text{ the characteristic function of the rationals}$$

$$\chi_C(x), \text{ the characteristic function of the Cantor set}$$

$$g = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Lebesgue Measure

Notation: Let $\mu(A)$ denote the Lebesgue measure of a set A.

Notation: Not all sets are measurable. Let $\mathcal{L}$ denote the collection of subsets of $\mathbb{R}$ which are measurable.

Density

Definition

Consider a sequence of intervals centered at a point x and shrinking around that point. We define the density using the measure of these intervals. Let $d_A(x)$ denote the density of a point x with respect to a set A:

$$d_A(x) = \lim_{n \to \infty} \frac{\mu\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap A\right)}{\mu\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right)}$$

provided the limit exists.

Example

1. Let $A = (0, 1)$. Every $x \in A$ has density $d_A(x) = 1$.
   *This holds for any open set A.

2. Consider $y \in A^c$. If $y \neq 0, 1$, then $d_A(y) = 0$. If $y = 0$ or 1, then $d_A(y) = \frac{1}{2}$.

3. Let $B = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$. Then for each $x \in B$, we have $d_B(x) = 0$. Actually, $d_{B^c}(x) = 1$.

Remark: Note that if a point has positive density in a set A, then A has positive measure.

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Density Function

Definition

The function $\Phi(A) : \mathcal{L}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is called the *density function* on $\mathbb{R}$ because it sends each set $A \in \mathcal{L}$ to the set of points with density 1 in A.

$$\Phi(A) = \{x \in \mathbb{R} : d_A(x) = 1\}$$

*There are extensions of Φ such that $\Phi(A)$ is defined for some non-measurable A. Such an extension is also called a density function, but is outside the scope of this presentation.

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Lebesgue Density Theorem

Theorem

Let $E \in \mathcal{L}$. Then $\mu(E \Delta \Phi(E)) = 0$. That is, for any measurable set, the symmetric difference of E and $\Phi(E)$ is a nullset.

Outline of Proof (Faure).

Consider the sets $A_n = \{x \in E \cap (-n, n) : d_E(x) < \frac{n}{n+1}\}$. Then as $n \to \infty$, the sets will expand to cover all of $\mathbb{R}$, and the density will approach 1. Notice that $E \setminus \Phi(E) = \bigcup_n A_n$; if we can show $\mu(A_n) = 0$, then $\mu(E \setminus \Phi(E)) = 0$.

Cover A_n with an open set U such that $\mu(U) < \mu(A_n) + \varepsilon$. It takes some proof to show that we can choose U such that $\mu(U) < n\varepsilon$ for any given ε. Since ε is arbitrary, A_n is a nullset. The countable union $\bigcup_{n=1}^{\infty} A_n$ is a nullset. □

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Approximate Continuity

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at x_0 iff there exists a set $A \in \mathcal{L}$ such that $x_0 \in \Phi(A)$ and

$$\lim_{x \to x_0,\ x \in A} f(x) = f(x_0)$$

Examples

1. Define $f(x) = \begin{cases} 1 & x = 1, \frac{1}{2}, \frac{1}{3}, \ldots \\ 0 & \text{otherwise} \end{cases}$.

   Then $f(x)$ is approximately continuous for all $x \neq 1, \frac{1}{2}, \frac{1}{3} \ldots$.

2. Consider $\chi_{\mathbb{Q}}$, the characteristic function of the rationals. $\chi_{\mathbb{Q}}(x) = 1$ if $x \in \mathbb{Q}$, 0 otherwise. $\chi_{\mathbb{Q}}$ is nowhere continuous, but it is approximately continuous for each $x \notin \mathbb{Q}$.

3. $$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

4. The collection of measurable functions is the same as the collection of approximately continuous functions.

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Density Topology

Definition

We define the *density topology* on $\mathbb{R}$ as follows: A set A is open in the density topology iff A is measurable and $A \subseteq \Phi(A)$, where $\Phi(A)$ is the set of density points of A. We denote the topology $(\mathbb{R}, \mathcal{T})$.

Remark: The union of measurable sets need not be measurable. It requires some proof to show that $A_i \in \mathcal{L}$ and A_i measurable $\Rightarrow \bigcup A_i \in \mathcal{L}$ and $\bigcup A_i \subseteq \Phi(\bigcup A_i)$.

Remark: A function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous iff f takes open sets in $\mathcal{T}$ to open sets in the usual topology.

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Examples

Set	$\Phi(\text{Set})$	Euclidean	Density Topology
$(0,1)$	$(0,1)$	open	open
$(0,1)\cup(1,2)$	$(0,2)$	open	open
$(-1,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$	$(-1,1)$	not open	open
$(0,1)\setminus\mathbb{Q}$	$(0,1)$	not open	open
$(0,1)\cup\{2\}$	$(0,1)$	not open	not open

Remark: The density topology is finer than the Euclidean topology on $\mathbb{R}$.

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Topological Properties

$(\mathbb{R}, \mathcal{T})$ **is a Hausdorff** (T_2) **space:** The density topology is finer than the usual topology.

$(\mathbb{R}, \mathcal{T})$ is not Separable

Theorem

The space $(\mathbb{R}, \mathcal{T})$ is not separable.

Proof by Contradiction.

Assume that $(\mathbb{R}, \mathcal{T})$ is separable. Then there exists a countable dense subset $V \subseteq \mathbb{R}$. Consider the set $U = \mathbb{R} \setminus V$. Since $\mu(V) = 0$, we have $U \subseteq \Phi(U)$. So U is open and V is not dense. □

$(\mathbb{R}, \mathcal{T})$ is not First Countable

Definition

A space is *first countable* if for each point x there exists a countable sequence of open neighborhoods of x: $N_1, N_2, \ldots$ such that given an open neighborhood U of x we can find an $N_i \subseteq U$ for some i.

Proof by Contradiction.

Assume that $(\mathbb{R}, \mathcal{T})$ is first countable. Fix $x \in \mathbb{R}$. Choose some $y_i \in N_i$ such that $y_i \neq x$ for each $i < \infty$. Then let $U = N_1 \setminus \{y_i : i < \infty\}$. Since $\mu(\{y_i\}) = 0$, we know that $U \subseteq \Phi(U)$ so U is open in $\mathcal{T}$. Then notice that $N_i \not\subseteq U$ for each i. □

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$(\mathbb{R}, \mathcal{T})$ is Baire

Lemma

A set A is a nullset iff A is nowhere dense in $\mathcal{T}$.

Proof.

Let A be a nullset. Then $\Phi(A^c) = \mathbb{R}$, so $\Phi(A) = \emptyset$. But $A^c \subseteq \Phi(A^c) = \mathbb{R}$, so A^c is an open set. Then A is closed. Also, $Int(A) \subseteq \Phi(A) \cap A = \emptyset$. So A is a closed set with empty interior, and A is nowhere dense.

Let A be nowhere dense. Then $\Phi(A) = \emptyset$. Using the Lebesgue Density Theorem: $\mu(A \Delta \Phi(A)) = 0$ gives us $\mu(A) = 0$. □

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A topological space is a *Baire space* if every union of countably many closed nowhere dense sets has empty interior.

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Further Study

How can we translate the density topology into other spaces?

What are the necessary and sufficient conditions for a measure space to support the density topology?

What does the density topology look like on the space $\mathcal{C}[0, 1]$?

References

Alessandro Andretta and Riccardo Camerlo. *Descriptive Set Theory of the Lebesgue Density Theorem.* www.arxiv.org/ abs/1105.3355v1. May 17, 2011.

Udayan B Darji. *Limits of Differentiable Functions*. American Mathematical Society, Vol. 124, No. 1, 1996.

Claude-Alain Faure. *A Short Proof of Lebesgue's Density Theorem*. American Mathematical Monthly, Vol. 109, No. 2, 2002. pp 194-196.

Casper Goffman, C.J. Neugebauer, T. Nishiura. *Density Topology and Approximate Continuity*. Duke Mathematical Journal, Vol 28, No. 4, 1961. pp 497-505.

John C. Oxtoby. *Measure and Category*, second edition (1980). Graduate Texts in Mathematics, Springer-Verlag, New York. 1980.

Dave L. Renfro. *Lebesgue Density Theorem Fails for Baire Category*. dave.renfro@act.org, Coralville, IA, 2014.

Franklin D. Tall. *The Density Topology*. Pacific Journal of Mathematics, Vol. 62, No. 1, 1976.

Wladislaw Wilczynski. *Density Topologies*. The Handbook of Set Theory, Vol. 1, Chp. 15, pp. 677-702. Elsevier Press, 2002.