Topological applications of long $\omega_1$-approximation sequences III

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Outline of a proof of $\text{Nt}(X) = \aleph_0$ where $h : 2^\lambda \to [0, 1]^\kappa$ is continuous, $X = h[2^\lambda]$, and $\pi_X(p, X) = w(X) = \kappa$ for all $p \in X$:

1. $\mathcal{A}$ is a base of $X$ of size $\kappa$ consisting of $F_\sigma$ sets.
2. $(M_\alpha)_{\alpha < \kappa}$ is a long $\omega_1$-approximation sequence with $h, \mathcal{A} \in M_0$.
3. $\mathcal{W}_\alpha \upharpoonright M_\alpha \subset \mathcal{A}_\alpha \upharpoonright M_\alpha$ is an efficient base of $X \upharpoonright M_\alpha$.
4. $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{< \alpha}$.
5. $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \; \overline{U} \subset V\}$.
6. $\mathcal{U} = \mathcal{U}_{< \kappa}$ is a base of $X$.
7. $h^{-1}[\overline{U}] \subset E_{\alpha, U}$ clopen $\subset \cap\{h^{-1}[W]: \overline{U} \subset W \in \mathcal{W}_\alpha\}$.
8. $\text{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{E_{\alpha, U} : U \in \mathcal{U}_\alpha\}$.

9. $\text{Nt}(\mathcal{D}) = \aleph_0$ where $\mathcal{D} = \mathcal{D}_{< \kappa}$.
10. $\text{Nt}(\mathcal{U}) = \aleph_0$. 
Let $\mathcal{B} = \text{Clop}(2^{\lambda})$.

Let $\mathcal{C} = \mathcal{B} \cap \uparrow\{h^{-1}[U] : U \in \mathcal{U}\}$.

Let $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$. Note that $\mathcal{D}_\alpha \subset \mathcal{C}_\alpha$.

To prove $\text{Nt}(\mathcal{D}) = \aleph_0$, it suffices to show that, for all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \cap \uparrow\mathcal{D}_\alpha$,

2. $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and

3. $H \uparrow \cap \mathcal{D}_\alpha = \emptyset$. 
For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,

(2) $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and

(3) $H \uparrow \cap \mathcal{D}_\alpha = \emptyset$:

To prove $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$, suppose that $K \in \mathcal{C}_\alpha$.

Then $M_\alpha$ knows that $h^{-1}[A] \subset K$ for some $A \in \mathcal{A}$.

So, choosing $A$ as above in $\mathcal{A}_\alpha$, we then find $U \subset W \subset A$ where $U \in \mathcal{U}_\alpha$ and $W \in \mathcal{W}_\alpha$, using the fact that $\mathcal{W}_\alpha \upharpoonright M_\alpha$ is a base and $\mathcal{U}_\alpha$ is a downward-closed subset of $\mathcal{W}_\alpha$.

We then have $\mathcal{D}_\alpha \ni E_{\alpha,U} \subset h^{-1}[W] \subset h^{-1}[A] \subset K$. 
For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
(2) $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and
(3) $H \uparrow \cap \mathcal{D}_\alpha = \emptyset$:

To prove $H \uparrow \cap \mathcal{D}_\alpha = \emptyset$, we suppose $H \subset E_{\alpha, U} \in \mathcal{D}_\alpha$ and deduce a contradiction.

By definition of $\mathcal{U}_\alpha$, we have $\overline{U} \subset V$ for some $V \in \mathcal{V}_\alpha$.

Inductively assuming $\mathcal{C}_{<\alpha} \subset \uparrow \mathcal{D}_{<\alpha}$, there exist $\beta < \alpha$ and $E_{\beta, T} \in \mathcal{D}_\beta$ such that $E_{\beta, T} \subset H$. Hence,

$$h^{-1}[T] \subset E_{\beta, T} \subset H \subset E_{\alpha, U} \subset h^{-1}[V].$$

Hence, $T \subset V$. But $T \in \mathcal{U}_\beta \subset \mathcal{W}_{<\alpha}$ and $V \in \mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$. Contradiction.
For all $\alpha < \kappa$ and $H \in \mathcal{C}_<\alpha$,

(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,

(2) $H \uparrow \cap \mathcal{D}_<\alpha$ is finite, and

(3) $H \uparrow \cap \mathcal{D}_\alpha = \emptyset$:

To prove that every $H \uparrow \cap \mathcal{D}_<\alpha$ is finite, proceed by induction on $\alpha$. (3) makes limit steps trivial.

Suppose that $K \in \mathcal{D}_{\alpha+1}$. We will show that $K \uparrow \cap \mathcal{D}_{<\alpha+1}$ is finite.

If $K \in \mathcal{D}_{<\alpha}$, then $K \uparrow \cap \mathcal{D}_{<\alpha+1}$ equals $K \uparrow \cap \mathcal{D}_{<\alpha}$, which is finite by our induction hypothesis.

So, assume that $K \in \mathcal{D}_\alpha$. Since $\text{Nt}(\mathcal{D}_\alpha) = \aleph_0$, the set $K \uparrow \cap \mathcal{D}_\alpha$ is finite.

Therefore, it suffices to show that $K \uparrow \cap \mathcal{D}_{<\alpha}$ is finite.

Recall that $\forall(\alpha)$ is finite, $M_{<\alpha} = \bigcup_{i \in \forall(\alpha)} N^i_{\alpha}$, and $N^i_{\alpha} < H(\theta)$. 
For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
2. $H^{\uparrow \cap \mathcal{D}_{<\alpha}}$ is finite, and
3. $H^{\uparrow \cap \mathcal{D}_\alpha} = \emptyset$:

It suffices to show that each $K^{\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha}$ is finite.

By our induction hypothesis, it suffices to find $H \in \mathcal{C}_{<\alpha}$ such that $K^{\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha} = H^{\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha}$.

Since $\mathcal{B}$ is just $\text{Clop}(2^\lambda)$, $H = \{ p \in 2^\lambda : p \upharpoonright N^i_\alpha \in K \upharpoonright N^i_\alpha \}$ satisfies $K \subset H \in \mathcal{B} \cap N^i_\alpha$ and $K^{\uparrow \cap \mathcal{B} \cap N^i_\alpha} = H^{\uparrow \cap \mathcal{B} \cap N^i_\alpha}$.

Since $K \in \mathcal{C}$ and $\mathcal{C}$ is upward closed in $\mathcal{B}$, we have $H \in \mathcal{C} \cap N^i_\alpha \subset \mathcal{C}_{<\alpha}$.

Since $\mathcal{D}_{<\alpha} \subset \mathcal{C}_{<\alpha} \subset \mathcal{B}$, we have $K^{\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha} = H^{\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha}$.
Outline of a proof of $\text{Nt}(X) = \aleph_0$ where $h: 2^\lambda \to [0, 1]^\kappa$ is continuous, $X = h[2^\lambda]$, and $\pi_X(p, X) = w(X) = \kappa$ for all $p \in X$:

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4. $\mathcal{V}_\alpha = \mathcal{W}_\alpha \upharpoonright \mathcal{W}_{< \alpha}$.
5. $\mathcal{U}_\alpha = \{ U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \; \overline{U} \subset V \}$.
6. $\mathcal{U} = \mathcal{U}_{< \kappa}$ is a base of $X$.
7. $h^{-1}[\overline{U}] \subset E_{\alpha, U}$ clopen $\subset \bigcap \{ h^{-1}[W] : \overline{U} \subset W \in \mathcal{W}_\alpha \}$.
8. $\text{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{ E_{\alpha, U} : U \in \mathcal{U}_\alpha \}$.
9. $\text{Nt}(\mathcal{D}) = \aleph_0$ where $\mathcal{D} = \mathcal{D}_{< \kappa}$.

10. $\text{Nt}(\mathcal{U}) = \aleph_0$. 
Seeking a contradiction, suppose that

\[ T \subset U_m \neq U_n \text{ and } T, U_m, U_n \in \mathcal{U} \text{ for all } m < n < \omega. \]

Let \( T \in \mathcal{U}_\alpha \) and let \( U_m \in \mathcal{U}_{\beta_m} \) for all \( m < \omega \).

Choose \( S \in \mathcal{U}_\alpha \) such that \( \overline{S} \subset T \). Then, for all \( m \), we have

\[ D \ni E_{\alpha,S} \subset h^{-1}[T] \subset h^{-1}[U_m] \subset E_{\beta_m,U_m} \in \mathcal{D}. \]

Since \( \text{Nt}(\mathcal{D}) = \aleph_0 \), we may thin out \((\beta_m)_{m<\omega}\) such that,

for some \( \beta < \kappa \) and \( U \in \mathcal{U}_\beta \), we have \( \forall m \ E_{\beta_m,U_m} = E_{\beta,U} \).

Thin out \((\beta_m)_{m<\omega}\) again to make it constant or strictly increasing.
In the case $\beta_0 < \beta_1$, we have $\overline{U_1} \subset V$ for some $V \in \mathcal{V}_{\beta_1}$, so
\[ h^{-1}[U_0] \subset E_{\beta,U} \subset h^{-1}[V], \]
in contradiction with $U_0 \in U_{\beta_0} \subset \mathcal{W}_{<\beta_1}$ and $V \in \mathcal{V}_{\beta_1} = \mathcal{W}_{\beta_1} \uparrow \mathcal{W}_{<\beta_1}$.

So, we are in the other case, $\beta_0 = \beta_m$ for all $m < \omega$.

Since $\mathcal{W}_{\beta_0} \uparrow M_{\beta_0}$ is an efficient base, each $U_m$ a finite set $\mathcal{F}_m$ of strict supersets in $\mathcal{W}_{\beta_0}$, but $\bigcup_{m<\omega} \mathcal{F}_m$ is infinite.

Given an arbitrary $i < \omega$, choose $j > i$ such that $\mathcal{F}_j \not\subseteq \mathcal{F}_i$.

Choose $W \in \mathcal{F}_j \setminus \mathcal{F}_i$. Since $\mathcal{W}_\alpha \uparrow M_\alpha$ is an efficient base, $\overline{U_j} \subset W$.

Hence, $h^{-1}[\overline{U_i}] \subset E_{\beta,U} \subset h^{-1}[W]$; hence, $\overline{U_i} \subset W$. But $\neg(U_i \not\subset W)$.

Hence $U_i = \overline{U_i} = W$; hence, $h^{-1}[U_i] = E_{\beta,U}$.

Thus, $U_i = h[E_{\beta,U}]$ for all $i < \omega$. Contradiction. $\square$
An *FN-map* on a boolean algebra $B$ is a function $f : B \to [B]^{<\aleph_0}$ such that, for all weakly increasing pairs $x \leq y$ in $B$, there exists $z \in f(x) \cap f(y)$ such that $x \leq z \leq y$.

$B$ has the Freese-Nation (FN) property if it has an FN map.

A boolean subalgebra $A$ of $B$ is *relatively complete* if, for every $b \in B$, there exists $a \in A$ such that $A \cap \uparrow b = A \cap \uparrow a$. In this case we write $A \leq_{rc} B$.

(Fuchino, 1994) The following are equivalent.

1. $B$ has the FN.
2. $B \cap M \leq_{rc} B$ for all countable $M \prec H(\theta)$ with $B \in M$.
3. $B \cap M \leq_{rc} B$ for all $M \prec H(\theta)$ with $B \in M$. 
(Fuchino, 1994) The following are equivalent.

(1) \( B \) has the FN.
(2) \( B \cap M \leq_{rc} B \) for all countable \( M \prec H(\theta) \) with \( B \in M \).
(3) \( B \cap M \leq_{rc} B \) for all \( M \prec H(\theta) \) with \( B \in M \).

Proof of (3) \( \Rightarrow \) (1) using a long \( \omega_1 \)-approximation sequence:

Let \( (M_\alpha)_{\alpha < |B|} \) be a long \( \omega_1 \)-approximation sequence with \( B \in M_0 \). For each \( x \in B \), let \( \rho(x) = \min\{\alpha : x \in M_\alpha\} \).

For each \( \alpha < |B| \), choose a well-ordering \( \sqsubseteq_\alpha \) of \( \{x \in B : \rho(x) = \alpha\} \) with length at most \( \omega \). Set \( \sqsubseteq = \bigcup_{\alpha < |A|} \sqsubseteq_\alpha \)

For each \( \alpha, i < \aleph(\alpha) \), and \( x \) with \( \alpha = \rho(x) \), since \( B \cap N_\alpha^i \leq_{rc} B \), there exist \( \pi^i_+(x) = \min(B \cap N_\alpha^i \cap \uparrow x) \) and \( \pi^i_-(x) = \max(B \cap N_\alpha^i \cap \downarrow x) \).

\( \rho(\pi^i_+(x)), \rho(\pi^i_-(x)) < \rho(x) \) for all \( i < \aleph(\alpha) \). (There is no \( i < \aleph(0) \).)
Recursively define $f : B \to [B]^{<\mathbb{N}_0}$ by

$$f(x) = \{y : y \sqsubseteq x\} \cup \bigcup_{i < \mathbb{N}(\rho(x))} \left(f(\pi_i^+(x)) \cup f(\pi_i^-(x))\right).$$

Suppose $x \leq y$. We verify that $S = [x,y] \cap f(x) \cap f(y)$ is nonempty by induction on $\max\{\rho(x), \rho(y)\}$.

If $\rho(x) = \rho(y)$, then

$x \sqsubseteq y$, in which case $x \in S$, or

$y \sqsubseteq x$, in which case $y \in S$.

If $\rho(x) < \rho(y)$, then $x \in N^i_{\rho(y)}$ for some $i$, in which case

$[x, \pi_i^-(y)] \cap f(x) \cap f(\pi_i^-(y))$ is a nonempty subset of $S$.

If $\rho(y) < \rho(x)$, then $y \in N^i_{\rho(x)}$ for some $i$, in which case

$[\pi_i^+(x), y] \cap f(\pi_i^+(x)) \cap f(y)$ is a nonempty subset of $S$. □
All free boolean algebras (\textit{i.e.}, algebras isomorphic to some $\text{Clop}(2^{\lambda})$) and their retracts (\textit{i.e.}, projective boolean algebras) have the FN.

All countable boolean algebras are retracts of $\text{Clop}(2^{\omega})$.

All $\aleph_1$-sized boolean algebras with the FN are retracts of $\text{Clop}(2^{\omega_1})$.

If $\kappa \geq \omega_2$, then the clopen algebra $\exp(\text{Clop}(2^{\omega_2}))$ of the Vietoris hyperspace $\exp(2^{\kappa})$ of nonempty closed subsets of $2^{\kappa}$ has the FN but is not a retract of a free boolean algebra and not even a subalgebra of a free boolean algebra.

Topologically speaking, $\exp(2^{\kappa})$ is openly generated but is not Dugundji and not even dyadic.

Our theorem about homogeneous dyadic compacta generalizes a bit:

If $X$ is a homogeneous continuous image of the Stone space $\text{Ult}(B)$ of a boolean algebra $B$ with the FN, then $\text{Nt}(X) = \aleph_0$. 
Two boolean subalgebras $A, B \subseteq C$ commute if, for all pairs $A \ni x \leq y \in B$, there exists $z \in A \cap B$ such that $x \leq z \leq y$.

(Heindorf–Shapiro, 1994)

- A boolean algebra has the strong Freese-Nation property (SFN) if it has a pairwise commuting cofinal family of finite subalgebras.
- Retracts of free boolean algebras have the SFN.
- $\exp(\text{Clop}(2^{\omega_2}))$ has SFN.
- The SFN implies the FN.
- Does the FN imply the SFN?

**Theorem** (Milovich, 2014). There is a boolean algebra of size $\aleph_2$ with the FN but not the SFN.

The proof uses a long $\omega_1$-approximation sequence and uses almost all of coherence properties mentioned in Part I.
Lajos Soukup has recently announced a $\sigma$-closed version of long $\omega_1$-approximation sequences:

Assume GCH and $\square^{**}_\mu$ for all regular uncountable $\mu$. Then, for every cardinal $\kappa$ and set $x$, there exist $(M_\alpha)_{\alpha<\kappa}$ and $(N_\alpha^i)_{i<\omega;\alpha<\kappa}$ such that

- $\kappa \subset \bigcup_{\alpha<\kappa} M_\alpha$.
- $x \in M_\alpha$,
- $|M_\alpha| = \aleph_1$,
- $M_{<\alpha} = \bigcup_{i<\omega} N_\alpha^i$,
- $[M_\alpha]^\omega \subset M_\alpha \prec H(\theta)$, and
- $[N_\alpha^i]^\omega \subset N_\alpha^i \prec H(\theta)$. 
References


