Topological applications of long ω_1-approximation sequences III

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Outline of a proof of $\mathrm{Nt}(X) = \aleph_0$ where $h\colon 2^\lambda \to [0,1]^\kappa$ is continuous, $X = h[2^\lambda]$, and $\pi\chi(p, X) = \mathrm{w}(X) = \kappa$ for all $p \in X$:

1. $\mathcal{A}$ is a base of X of size κ consisting of F_σ sets.
2. $(M_\alpha)_{\alpha<\kappa}$ is a long ω_1-approximation sequence with $h, \mathcal{A} \in M_0$.
3. $\mathcal{W}_\alpha \restriction M_\alpha \subset \mathcal{A}_\alpha \restriction M_\alpha$ is an efficient base of $X \restriction M_\alpha$.
4. $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$.
5. $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha\ \overline{U} \subset V\}$.
6. $\mathcal{U} = \mathcal{U}_{<\kappa}$ is a base of X.
7. $h^{-1}[\overline{U}] \subset E_{\alpha,U}$ clopen $\subset \bigcap\{h^{-1}[W] : \overline{U} \subset W \in \mathcal{W}_\alpha\}$.
8. $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$.

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9. $\mathrm{Nt}(\mathcal{D}) = \aleph_0$ where $\mathcal{D} = \mathcal{D}_{<\kappa}$.
10. $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

Let $\mathcal{B} = \text{Clop}(2^\lambda)$.

Let $\mathcal{C} = \mathcal{B} \cap \uparrow\{h^{-1}[U] : U \in \mathcal{U}\}$.

Let $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$. Note that $\mathcal{D}_\alpha \subset \mathcal{C}_\alpha$.

To prove $\text{Nt}(\mathcal{D}) = \aleph_0$, it suffices to show that, for all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \subset \uparrow\mathcal{D}_\alpha$,
2. $H\uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and
3. $H\uparrow \cap \mathcal{D}_\alpha = \varnothing$.

For all $\alpha < \kappa$ *and* $H \in \mathcal{C}_{<\alpha}$*,*
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$*,*
(2) $H\uparrow \cap \mathcal{D}_{<\alpha}$ *is finite, and*
(3) $H\uparrow \cap \mathcal{D}_\alpha = \varnothing$*:*

To prove $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$, suppose that $K \in \mathcal{C}_\alpha$.

Then M_α knows that $h^{-1}[A] \subset K$ for some $A \in \mathcal{A}$.

So, choosing A as above in $\mathcal{A}_\alpha$, we then find $\overline{U} \subset W \subset A$ where $U \in \mathcal{U}_\alpha$ and $W \in \mathcal{W}_\alpha$, using the fact that $\mathcal{W}_\alpha \restriction M_\alpha$ is a base and $\mathcal{U}_\alpha$ is a downward-closed subset of $\mathcal{W}_\alpha$.

We then have $\mathcal{D}_\alpha \ni E_{\alpha,U} \subset h^{-1}[W] \subset h^{-1}[A] \subset K$.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,
(2) $H \uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and
(3) *$H \uparrow \cap \mathcal{D}_\alpha = \varnothing$:*

To prove $H \uparrow \cap \mathcal{D}_\alpha = \varnothing$, we suppose $H \subset E_{\alpha,U} \in \mathcal{D}_\alpha$ and deduce a contradiction.

By definition of $\mathcal{U}_\alpha$, we have $\overline{U} \subset V$ for some $V \in \mathcal{V}_\alpha$.

Inductively assuming $\mathcal{C}_{<\alpha} \subset \uparrow \mathcal{D}_{<\alpha}$, there exist $\beta < \alpha$ and $E_{\beta,T} \in \mathcal{D}_\beta$ such that $E_{\beta,T} \subset H$. Hence,

$$h^{-1}[T] \subset E_{\beta,T} \subset H \subset E_{\alpha,U} \subset h^{-1}[V].$$

Hence, $T \subset V$. But $T \in \mathcal{U}_\beta \subset \mathcal{W}_{<\alpha}$ and $V \in \mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$. Contradiction.

For all $\alpha < \kappa$ *and* $H \in \mathcal{C}_{<\alpha}$*,*
(1) $\mathcal{C}_\alpha \subset \uparrow\mathcal{D}_\alpha$*,*
(2) $H\uparrow\cap\mathcal{D}_{<\alpha}$ *is finite, and*
(3) $H\uparrow\cap\mathcal{D}_\alpha = \varnothing$*:*

To prove that every $H\uparrow\cap\mathcal{D}_{<\alpha}$ is finite, proceed by induction on α. (3) makes limit steps trivial.

Suppose that $K \in \mathcal{D}_{<\alpha+1}$. We will show that $K\uparrow\cap\mathcal{D}_{<\alpha+1}$ is finite.

If $K \in \mathcal{D}_{<\alpha}$, then $K\uparrow\cap\mathcal{D}_{<\alpha+1}$ equals $K\uparrow\cap\mathcal{D}_{<\alpha}$, which is finite by our induction hypothesis.

So, assume that $K \in \mathcal{D}_\alpha$. Since $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$, the set $K\uparrow\cap\mathcal{D}_\alpha$ is finite.

Therefore, it suffices to show that $K\uparrow\cap\mathcal{D}_{<\alpha}$ is finite.

Recall that $\daleth(\alpha)$ is finite, $M_{<\alpha} = \bigcup_{i\in\daleth(\alpha)} N^i_\alpha$, and $N^i_\alpha \prec H(\theta)$.

For all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,
(1) $\mathcal{C}_\alpha \subset \uparrow\mathcal{D}_\alpha$,
(2) *$H\uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and*
(3) $H\uparrow \cap \mathcal{D}_\alpha = \varnothing$:

It suffices to show that each $K\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$ is finite.

By our induction hypothesis, it suffices to find $H \in \mathcal{C}_{<\alpha}$ such that $K\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha = H\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$.

Since $\mathcal{B}$ is just Clop(2^λ), $H = \{p \in 2^\lambda : p \restriction N^i_\alpha \in K \restriction N^i_\alpha\}$ satisfies $K \subset H \in \mathcal{B} \cap N^i_\alpha$ and $K\uparrow \cap \mathcal{B} \cap N^i_\alpha = H\uparrow \cap \mathcal{B} \cap N^i_\alpha$.

Since $K \in \mathcal{C}$ and $\mathcal{C}$ is upward closed in $\mathcal{B}$, we have $H \in \mathcal{C} \cap N^i_\alpha \subset \mathcal{C}_{<\alpha}$.

Since $\mathcal{D}_{<\alpha} \subset \mathcal{C}_{<\alpha} \subset \mathcal{B}$, we have $K\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha = H\uparrow \cap \mathcal{D}_{<\alpha} \cap N^i_\alpha$.

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8. $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ where $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$.
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10. $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

Seeking a contradiction, suppose that

$T \subset U_m \neq U_n$ and $T, U_m, U_n \in \mathcal{U}$ for all $m < n < \omega$.

Let $T \in \mathcal{U}_\alpha$ and let $U_m \in \mathcal{U}_{\beta_m}$ for all $m < \omega$.

Choose $S \in \mathcal{U}_\alpha$ such that $\overline{S} \subset T$. Then, for all m, we have

$$\mathcal{D} \ni E_{\alpha,S} \subset h^{-1}[T] \subset h^{-1}[U_m] \subset E_{\beta_m,U_m} \in \mathcal{D}.$$

Since $\mathrm{Nt}(\mathcal{D}) = \aleph_0$, we may thin out $(\beta_m)_{m<\omega}$ such that,

for some $\beta < \kappa$ and $U \in \mathcal{U}_\beta$, we have $\forall m\ E_{\beta_m,U_m} = E_{\beta,U}$.

Thin out $(\beta_m)_{m<\omega}$ again to make it constant or strictly increasing.

In the case $\beta_0 < \beta_1$, we have $\overline{U_1} \subset V$ for some $V \in \mathcal{V}_{\beta_1}$, so

$$h^{-1}[U_0] \subset E_{\beta,U} \subset h^{-1}[V],$$

in contradiction with $U_0 \in \mathcal{U}_{\beta_0} \subset \mathcal{W}_{<\beta_1}$ and $V \in \mathcal{V}_{\beta_1} = \mathcal{W}_{\beta_1} \setminus \uparrow\mathcal{W}_{<\beta_1}$.

So, we are in the other case, $\beta_0 = \beta_m$ for all $m < \omega$.

Since $\mathcal{W}_{\beta_0} \restriction M_{\beta_0}$ is an efficient base, each U_m a finite set $\mathcal{F}_m$ of strict supersets in $\mathcal{W}_{\beta_0}$, but $\bigcup_{m<\omega} \mathcal{F}_m$ is infinite.

Given an arbitrary $i < \omega$, choose $j > i$ such that $\mathcal{F}_j \not\subseteq \mathcal{F}_i$.

Choose $W \in \mathcal{F}_j \setminus \mathcal{F}_i$. Since $\mathcal{W}_\alpha \restriction M_\alpha$ is an efficient base, $\overline{U_j} \subset W$.

Hence, $h^{-1}[\overline{U_i}] \subset E_{\beta,U} \subset h^{-1}[W]$; hence, $\overline{U_i} \subset W$. But $\neg(U_i \subsetneq W)$.

Hence $U_i = \overline{U_i} = W$; hence, $h^{-1}[U_i] = E_{\beta,U}$.

Thus, $U_i = h[E_{\beta,U}]$ for all $i < \omega$. Contradiction. $\square$

An *FN-map* on a boolean algebra B is a function $f\colon B \to [B]^{<\aleph_0}$ such that, for all weakly increasing pairs $x \leq y$ in B, there exists $z \in f(x) \cap f(y)$ such that $x \leq z \leq y$.

B has the Freese-Nation (FN) property if it has an FN map.

A boolean subalgebra A of B is *relatively complete* if, for every $b \in B$, there exists $a \in A$ such that $A \cap \uparrow b = A \cap \uparrow a$. In this case we write $A \leq_{\mathrm{rc}} B$.

(Fuchino, 1994) The following are equivalent.
(1) B has the FN.
(2) $B \cap M \leq_{\mathrm{rc}} B$ for all countable $M \prec H(\theta)$ with $B \in M$.
(3) $B \cap M \leq_{\mathrm{rc}} B$ for all $M \prec H(\theta)$ with $B \in M$.

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(3) $B \cap M \leq_{\mathsf{rc}} B$ for all $M \prec H(\theta)$ with $B \in M$.

Proof of (3)$\Rightarrow$(1) using a long ω_1-approximation sequence:

Let $(M_\alpha)_{\alpha<|B|}$ be a long ω_1-approximation sequence with $B \in M_0$. For each $x \in B$, let $\rho(x) = \min\{\alpha : x \in M_\alpha\}$.

For each $\alpha < |B|$, choose a well-ordering $\sqsubseteq_\alpha$ of $\{x \in B : \rho(x) = \alpha\}$ with length at most ω. Set $\sqsubseteq = \bigcup_{\alpha<|A|} \sqsubseteq_\alpha$

For each α, $i < \daleth(\alpha)$, and x with $\alpha = \rho(x)$, since $B \cap N^i_\alpha \leq_{\mathsf{rc}} B$, there exist $\pi^i_+(x) = \min(B \cap N^i_\alpha \cap \uparrow x)$ and $\pi^i_-(x) = \max(B \cap N^i_\alpha \cap \downarrow x)$.

$\rho(\pi^i_+(x)), \rho(\pi^i_-(x)) < \rho(x)$ for all $i < \daleth(\alpha)$. (There is no $i < \daleth(0)$.)

Recursively define $f\colon B \to [B]^{<\aleph_0}$ by

$$f(x) = \{y : y \sqsubseteq x\} \cup \bigcup_{i<\daleth(\rho(x))} \left(f(\pi^i_+(x)) \cup f(\pi^i_-(x))\right).$$

Suppose $x \leq y$. We verify that $S = [x, y] \cap f(x) \cap f(y)$ is nonempty by induction on $\max\{\rho(x), \rho(y)\}$.

If $\rho(x) = \rho(y)$, then
$x \sqsubseteq y$, in which case $x \in S$, or
$y \sqsubseteq x$, in which case $y \in S$.

If $\rho(x) < \rho(y)$, then $x \in N^i_{\rho(y)}$ for some i, in which case $[x, \pi^i_-(y)] \cap f(x) \cap f(\pi^i_-(y))$ is a nonempty subset of S.

If $\rho(y) < \rho(x)$, then $y \in N^i_{\rho(x)}$ for some i, in which case $[\pi^i_+(x), y] \cap f(\pi^i_+(x)) \cap f(y)$ is a nonempty subset of S. $\square$

All free boolean algebras (*i.e.*, algebras isomorphic to some Clop(2^λ)) and their retracts (*i.e.*, projective boolean algebras) have the FN.

All countable boolean algebras are retracts of Clop(2^ω).

All $\aleph_1$-sized boolean algebras with the FN are retracts of Clop(2^{ω_1}).

If $\kappa \geq \omega_2$, then the clopen algebra exp(Clop(2^{ω_2})) of the Vietoris hyperspace exp(2^κ) of nonempty closed subsets of 2^κ has the FN but is not a retract of a free boolean algebra and not even a subalgebra of a free boolean algebra.

Topologically speaking, exp(2^κ) is openly generated but is not Dugundji and not even dyadic.

Our theorem about homogeneous dyadic compacta generalizes a bit:

If X is a homogeneous continuous image of the Stone space Ult(B) of a boolean algebra B with the FN, then Nt(X) $= \aleph_0$.

Two boolean subalgebras $A, B \subset C$ *commute* if, for all pairs $A \ni x \leq y \in B$, there exists $z \in A \cap B$ such that $x \leq z \leq y$.

(Heindorf–Shapiro, 1994)

- A boolean algebra has the *strong Freese-Nation property* (SFN) if it has a pairwise commuting cofinal family of finite subalgebras.
- Retracts of free boolean algebras have the SFN.
- exp(Clop(2^{ω_2})) has SFN.
- The SFN implies the FN.
- Does the FN imply the SFN?

Theorem (Milovich, 2014). There is a boolean algebra of size $\aleph_2$ with the FN but not the SFN.

The proof uses a long ω_1-approximation sequence and uses almost all of coherence properties mentioned in Part I.

Lajos Soukup has recently announced a σ-closed version of long ω_1-approximation sequences:

Assume GCH and $\square^{**}_{\mu}$ for all regular uncountable μ. Then, for every cardinal κ and set x, there exist $(M_\alpha)_{\alpha<\kappa}$ and $(N^i_\alpha)_{i<\omega;\alpha<\kappa}$ such that

- $\kappa \subset \bigcup_{\alpha<\kappa} M_\alpha$.
- $x \in M_\alpha$,
- $|M_\alpha| = \aleph_1$,
- $M_{<\alpha} = \bigcup_{i<\omega} N^i_\alpha$,
- $[M_\alpha]^\omega \subset M_\alpha \prec H(\theta)$, and
- $[N^i_\alpha]^\omega \subset N^i_\alpha \prec H(\theta)$.

References

L. Heindorf and L. B. Shapiro, *Nearly Projective Boolean Algebras*, with an appendix by S. Fuchino, Lecture Notes in Mathematics **1596**, Springer-Verlag, Berlin, 1994.

D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, *Topology and its Applications* **156** (2008), 443–464.

D. Milovich, *On the strong Freese-Nation property* (2014), arXiv:1412.7443.

D. Soukup, *Davies-trees in infinite combinatorics* (2014), arXiv:1407.3604.

L. Soukup, *On properties of families of sets (Part 3)* (2014), `http://bcc.impan.pl/14Young/index.php/slides`.