On a new $F_\sigma$ ideal

Adam Kwela

Institute of Mathematics, Polish Academy of Sciences

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An ideal $\mathcal{I}$ on $\omega$ is $\text{Mon}$ if every sequence of reals contains a monotone subsequence indexed by an $\mathcal{I}$-positive set. An ideal $\mathcal{I}$ is $k$-Ramsey if every coloring of $[\omega]^2$ by $k$ colors has a homogeneous $\mathcal{I}$-positive set.

$$\text{Ramsey} \Rightarrow \text{Mon}.$$

Filipów, Mrożek, Recław and Szuca asked if there is a $\text{Mon}$ ideal which is not $k$-Ramsey for some $k$? This question was answered by Meza-Alcántara, who showed the existence of a 2-Ramsey (so $\text{Mon}$) ideal, which is not 3-Ramsey. But we can reformulate this question:

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*Is there a Mon ideal which is not 2-Ramsey?*
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*Is there a Mon ideal which is not 2-Ramsey?*
Define a coloring $\chi: [\omega \times \omega]^2 \rightarrow \{\text{blue, red}\}$ by:

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\chi((i,j),(k,l)) = \begin{cases} 
\text{blue} & \text{if } k > i + j \\
\text{red} & \text{if } k \leq i + j
\end{cases}
$$

for $(i,j),(k,l) \in \omega \times \omega$ such that $(i,j) \leq_{\text{lex}} (k,l)$.

**Definition (K.)**

$\mathcal{K}$ is the ideal generated by $\chi$-homogeneous subsets of $\omega \times \omega$, i.e., sets $H \subset \omega \times \omega$ such that $\chi \upharpoonright [H]^2$ is constant.

It is immediate that $\mathcal{K}$ is not 2-Ramsey. Moreover one can prove that $\mathcal{K}$ is $F_\sigma$. 

A. Kwela

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\(\mathcal{K}\) is the ideal generated by \(\chi\)-homogeneous subsets of \(\omega \times \omega\), i.e., sets \(H \subset \omega \times \omega\) such that \(\chi \upharpoonright [H]^2\) is constant.

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The ideal $\mathcal{K}$ - some pictures
$\mathcal{K}$ is generated by two kinds of sets.

- All vertical lines, i.e., all sets $\{i\} \times \omega$ for $i \in \omega$;
- Subsets of $\omega \times \omega$ of the following form:

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[->, thick, black] (0,0) -- (9,0);
\draw[->, thick, black] (0,0) -- (0,9);
\fill[black] (0,0) circle (5pt);
\fill[black] (1,0) circle (5pt);
\fill[black] (2,0) circle (5pt);
\fill[black] (3,0) circle (5pt);
\fill[black] (4,0) circle (5pt);
\fill[black] (5,0) circle (5pt);
\fill[black] (6,0) circle (5pt);
\fill[black] (7,0) circle (5pt);
\fill[black] (8,0) circle (5pt);
\fill[black] (9,0) circle (5pt);
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Is there a Mon ideal which is not 2-Ramsey?

Theorem (K.)

Every ideal on $\omega$ isomorphic to $\mathcal{K}$ is Mon.

Corollary

$\mathcal{K}$ solves the Problem of Filipów, Mrożek, Recław and Szuca!
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Answer to the question of Filipów et al.

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A sequence \((x_i)_{i \in \omega}\) of reals is \(\mathcal{I}\)-convergent to \(x \in \mathbb{R}\) if \(\{i \in \omega : |x_i - x| \geq \epsilon\} \in \mathcal{I}\) for every \(\epsilon > 0\).

A function \(f : \mathbb{R} \to \mathbb{R}\) is a pointwise limit relatively to \(\mathcal{I}\) of a sequence of functions \((f_i)_{i \in \omega}\) if \((f_i(x))_{i \in \omega}\) is \(\mathcal{I}\)-convergent to \(f(x)\) for every \(x \in \mathbb{R}\).

For a family \(\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}\) by \(\text{LIM}(\mathcal{F})\) we denote the family of all functions which can be represented as a pointwise limit of a sequence of functions from \(\mathcal{F}\) (for instance, if \(C\) denotes the family of continuous functions then \(\text{LIM}(C)\) is the first Baire class).

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A (long) digression about ideal convergence - notations

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A (long) digression about ideal convergence

Theorem (Laczkovich and Recław, 2009)

Let $\mathcal{I}$ be a Borel ideal. TFAE:

1. $\mathcal{I}$-LIM$(C) = \text{LIM}(C)$;
2. $\mathcal{I}$ is a weak $P$-ideal;
3. $\text{Fin} \otimes \text{Fin} \not\leq_K \mathcal{I}$;
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$I$ is a weak $P$-ideal if for every $(X_i)_{i \in \omega} \subset I$ there is $X \notin I$ with $X \cap X_i$ finite for all $i$.

$I \leq_K J$ if there is $f : \bigcup J \to \bigcup I$ such that $f^{-1}[A] \in J$ for all $A \in I$. $I \sqsubseteq J$ if $f$ is a bijection.
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2. \( \mathcal{I} \) is a weak \( P \)-ideal;
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\( \mathcal{I} \) is a weak \( P \)-ideal if for every \( (X_i)_{i \in \omega} \subset \mathcal{I} \) there is \( X \notin \mathcal{I} \) with \( X \cap X_i \) finite for all \( i \).

\( \mathcal{I} \leq_k \mathcal{J} \) if there is \( f : \bigcup \mathcal{J} \to \bigcup \mathcal{I} \) such that \( f^{-1}[A] \in \mathcal{J} \) for all \( A \in \mathcal{I} \).

\( \mathcal{I} \sqsubseteq \mathcal{J} \) if \( f \) is a bijection.
Quasi-continuity

\( f : \mathbb{R} \rightarrow \mathbb{R} \) is quasi-continuous (\( f \in QC \)) if for every \( \epsilon > 0 \), \( x_0 \in \mathbb{R} \) and open neighborhood \( U \ni x_0 \) there is a nonempty open \( V \subset U \) such that \( |f(x) - f(x_0)| < \epsilon \) for all \( x \in V \).

Quasi-continuous are all continuous functions as well as all left-continuous (right-continuous) functions.

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Theorem (Natkaniec and Szuca, preprint)

Let $\mathcal{I}$ be a Borel ideal. TFAE:

1. $\mathcal{I}$-LIM$(\mathcal{QC}) = \text{LIM}(\mathcal{QC})$;
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Following Laflamme we call $\mathcal{I}$ weakly Ramsey if every tree $T \subset [\omega]^{<\omega}$ with $\{ n : s \upharpoonright n \in T \}$ in the dual filter for all $s \in T$, contains an $\mathcal{I}$-positive branch.

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Is there a counterpart of $\text{Fin} \otimes \text{Fin}$ for the above theorem?
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Another application of the ideal \( \mathcal{K} \)

**Theorem (K.)**

Let \( \mathcal{I} \) be any ideal. TFAE:

1. \( \mathcal{I} \) is not weakly Ramsey;
2. \( \mathcal{K} \leq_{\mathcal{K}} \mathcal{I} \);
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Let \( \mathcal{I} \) be a Borel ideal. TFAE:

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A. Kwela
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More about weakly Ramsey ideals

Recall that $\mathcal{I}$ is locally selective, if every partition $(X_n)_{n \in \omega} \subset \mathcal{I}$ has an $\mathcal{I}$-positive selector.

$\mathcal{I}$ is weakly selective, if every partition $(X_n)_{n \in \omega}$ with at most one element not in $\mathcal{I}$ and such that $\bigcup_{m \geq n} X_m \notin \mathcal{I}$ for each $n$, has an $\mathcal{I}$-positive selector.

$$\text{weakly selective } \Rightarrow \text{ locally selective}$$

Proposition (Essentially Grigorieff, 1971)

$\mathcal{I}$ is weakly Ramsey if and only if for every partition $(X_n)_{n \in \omega} \subset \mathcal{I}$, there exists a strictly increasing function $f : \omega \rightarrow \omega$, with $f[\omega] \notin \mathcal{I}$ and such that $f(n + 1) \in \bigcup_{i > f(n)} X_i$ for each $n \in \omega$. 
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For instance the ideal $(\emptyset \otimes Fin) \oplus (Fin \otimes Fin)$ is weakly Ramsey, but not weakly selective. On the other hand $\mathcal{K}$ is locally selective but not weakly Ramsey.
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Thank you for your attention!