# Embeddability properties of $\sigma$-discrete metrizable spaces

by

**Marta Walczyńska** (speaker) and **Szymon Plewik** (co-author)

(*Katowice*)

Winter School in Abstract Analysis section Set Theory & Topology
25th Jan — 1st Feb 2014

## Our motivations

In [Topology Appl. 148 (2005)], W. D. Gillam described (in detail) dimensional types of countable metrizable spaces. The purpose of our study is to generalize the topics of this paper to a wider class of metrizable spaces. We have gathered the facts contained in several articles and textbooks. Based on these selections, we are going to write a review article. This article will be part of my doctoral dissertation.

Generalization involves the replacement of countable metric spaces with $\sigma$-discrete metrizable spaces.

**Definition**

A metrizable space is called $\sigma$-discrete, if it is an union of countably many discrete subspaces.

# On Gillam's paper

For subspaces $X$ and $Y$ of the space $\mathbb{Q}$ of rational numbers, the symbol $X \leq_h Y$ means that $X$ is homeomorphic to a subspace of $Y$. Let $P(\mathbb{Q})$ be the family of all subsets of $\mathbb{Q}$. In fact, W. D. Gillam (2005) precisely described the poset $(P(\mathbb{Q}), \leq_h)$.

Let's make a few remarks:

– First time the relation $\leq_h$ was investigated by M. Fréchet (1910).
– Mazurkiewicz and Sierpiński (1920) proved: *A countable compact metric space $X$ is homeomorphic with the ordinal number $\omega^\alpha n + 1$, where $n = |X^{(\alpha)}|$ is a natural number and $X^{(\alpha)}$ is the $\alpha$-derivative of $X$ and the countable ordinal number $\omega^\alpha n + 1$ is equipped with the linear topology.*
– Knaster and Urbanik (1953) proved: *A countable scattered metric space has a scattered metrizable compactification.*
– The paper by Gillam (2005) does not cite papers by Mazurkiewicz and Sierpiński (1920) or by Knaster and Urbanik (1953) and treats their results like a folklore.

# Some properties of $\sigma$-discrete metrizable spaces

Any countable metrizable space, being countable sum of single points, is $\sigma$-discrete. In particular, the space $\mathbb{Q}$ of all rational numbers is $\sigma$-discrete.

Let $B(\mathfrak{m}) = \mathfrak{m}^\omega$ be the Baire space of weight $\mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal. Compare Engelking's *General Topology* p. 326. Put
$C(\mathfrak{m}) = \{y \in B(\mathfrak{m}) : \text{almost all coordinates of } y \text{ are equal to } 0\}$.

**See T. Przymusiński [Fund. Math. (1979)], compare Sz. Plewik [Colloq. Math. (1978)]**

- A nonempty metrizable $\sigma$-discrete space, with all nonempty open subsets of weight $\mathfrak{m}$, is homeomorphic to $C(\mathfrak{m})$.
- A metrizable $\sigma$-discrete space of the weight $\mathfrak{m}$ is homeomorphic to a subspace of $C(\mathfrak{m})$.

Such a characterization of rational numbers, where $\mathfrak{m} = \omega_0$, is assigned to Cantor, Brouwer or Sierpiński.

# Mertizable scattered spaces

We could not find (literally) the following fact.

**Theorem 1**

Any metrizable scattered space is $\sigma$-discrete.

Let us present a sketch of a proof based on the facts discussed in textbooks. Scattered means: has no dense in itself subspace.

(1). *A metrizable $\sigma$-discrete space is an union of countable many closed and discrete subspaces*. Hint: apply the Bing metrization theorem 4.4.8 from Engelking's *General Topology*.

(2). *A metrizable locally $\sigma$-discrete space is $\sigma$-discrete*. Hint: apply the Stone theorem 4.4.1 from Engelking's *General Topology*.

(3). *If a metrizable space $X$ is not $\sigma$-discrete, then*

$$\{x \in X : \text{ no neighborhood of } x \text{ is } \sigma\text{-discrete}\}$$

*is dense in itself.*

# Metrizable $\sigma$-discrete space is 0-dimensional

We need the following fact. Its simplified versions are applied in articles by Kannan and Rajagopalan (1974), Arosio and Ferreira (1980) and Telgársky (1968).

**Theorem 2**

Every open cover of a metrizable $\sigma$-discrete space has a refinement consisting of closed-open and pairwise disjoint sets.

Again, we could not find this fact, in the literature. Note that, for similar facts it is applied phrase "Every finite open cover of ..." in textbooks on dimension theory, for example in Engelking's *Teoria Wymiaru* or *General Topology*.

# Two facts needed for proof

Let us present a sketch of a proof of the theorem 2 based on the facts discussed in textbooks.

(4). *If a normal space is an union of countably many closed and discrete subspaces, then it has a base consisting of closed-open sets*. Hint: modify Engelking's reasoning 1.3.2 from *Teoria wymiaru*.

(5). A needed refinement one can construct inductively, using paracompactness. In fact, again using the Stone theorem. □

# The most important trick with $\alpha$-th derivative

For a space $X$ the $\alpha$-th derivative (denote by $X^{(\alpha)}$) is defined inductively:

$$X^{(0)} = X,$$
$$X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is not isolated in } X^{(\alpha)}\},$$
$$X^{(\alpha)} = \bigcap\{X^{(\beta)} : \beta < \alpha\} \text{ for a limit } \alpha.$$

Thus each $X^{(\alpha)}$ is closed. Suppose $X^{(\alpha)}$ is a discrete subspace, then we have two types of open covers:

$$\{X \setminus X^{(\beta)} : \beta < \alpha\}, \text{ whenever } X^{(\alpha)} = \emptyset;$$
$$\{X \setminus X^{(\alpha)}\} \cup \{V_x \subseteq X : V_x \cap X^{(\alpha)} = \{x\} \text{ and } x \in X^{(\alpha)}\}.$$

In both cases, these coverings typically are infinite. But typically, for the inductive proof, we need a refinement consisting of closed-open and pairwise disjoint sets. A such trick is based on a proof by Kannan and Rajagopalan (1974).

Now, assume that $X$ is a compact countable metric space. The relation $\cong$ means that spaces are homeomorphic.

- If $X^{(1)} = \emptyset$, then $X$ is finite. So, $X$ is homeomorphic to the ordinal number $\omega^0 \cdot |X| = 1 \cdot |X|$ (denote by $X \cong \omega^0 \cdot |X|$).
- If $|X^{(1)}| = n \in \omega$, then $X$ is the sum of $n$ convergent sequences, $X \cong G_1 \oplus G_2 \oplus \cdots \oplus G_n$, where each $G_i$ is a convergent sequence. So, $X \cong \omega \cdot n + 1$.
- If $|X^{(\alpha)}| = n \in \omega$, then $X$ has a finite open cover $\mathcal{U}$ such that any $V \in \mathcal{U}$ meets $X^{(\alpha)}$ at a single point. Then $X$ is the sum of $n$ subspaces, each one is homeomorphic to $\omega^\alpha + 1$. So $X \cong (\omega^\alpha + 1) \oplus (\omega^\alpha + 1) \oplus \cdots \oplus (\omega^\alpha + 1)$ and $X \cong \omega^\alpha \cdot n + 1$;

# Mazurkiewicz-Sierpiński theorem: Essential step

- If $|X^{(\alpha)}| = 1$, then $X \setminus X^{(\alpha)}$ is the sum of $\omega$-many closed-open subsets, each one has empty $\alpha$-derivative . By the induction conditions

$$X \setminus X^{(\alpha)} \cong (\omega^{\beta_0} \cdot n_0 + 1) \oplus (\omega^{\beta_1} \cdot n_1 + 1) \oplus \dots .$$

  If $\alpha = \gamma + 1$, then one can assume that every $\beta_n = \gamma$. If $\alpha$ is a limit ordinal, then every $\beta_n < \alpha$ and $\lim_{n\to\infty} \beta_n = \alpha$. In both cases we obtain $X \cong \omega^\alpha + 1$.

So, if you know how to add ordinal numbers, then the rest of the proof is a standard transfinite induction. In fact, for our purposes we need to slightly modify and complicate the original proof by Mazurkiewicz and Sierpiński.

# Final remarks

In induction steps, there are two types of cover:

$$\{X \setminus X^{(\beta)} : \beta < \alpha\};$$
$$\{X \setminus X^{(\alpha)}\} \cup \{V_x \subseteq X : V_x \cap X^{(\alpha)} = \{x\} \text{ and } x \in X^{(\alpha)}\}.$$

In both cases, we consider a refinement which consists of closed-open sets, each one (usually) with the indicated condensation point. We put these closed-open sets among the ordinals such that the sum topology coincides with the topology inherited from the order topology. Complexity of induction steps is the following:

- The use of convergent sequences is enough for the Mazurkiewicz-Sierpiński theorem;
- For the Knaster-Urbanik theorem we additionally need to consider closed-open subsets in which the indicated condensation point has neighborhoods containing infinite (countable) closed and discrete subsets.

- For scattered metrizable spaces we additionally need to consider closed-open subsets in which the indicated condensation point has neighborhoods containg infinite (countable or uncountable) closed and discrete subsets.
- For $\sigma$-discrete metrizable spaces we additionally consider spaces $C(\mathfrak{m})$ for uncountable $\mathfrak{m}$.

- A. Arosio, A. V. Ferreira, *On nonseparable 0-dimensional metrizable spaces*. Portugal. Math. 37 (1978), no. 3-4, 273–297 (1981).
- R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw, (1977).
- R. Engelking, *Teoria wymiaru*. (Państwowe Wydawnictwo Naukowe, Warsaw, (1977).
- M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann., 68 (1910), 145 - 168.
- W. D. Gillam, *Embeddability properties of countable metric spaces*. Topology Appl. 148 (2005), no. 1-3, 63-82.
- V. Kannan, M. Rajagopalan, *On scattered spaces*. Proc. Amer. Math. Soc. 43 (1974), 402–408.

- B. Knaster, K. Urbanik, *Sur les espaces complets séparables de dimension 0*. Fund. Math. 40, (1953). 194–202.
- S. Mazurkiewicz, W. Sierpiński, *Contribution à la topologie des ensembles dénombrables* . Fund. Math. I (1920), 17-27.
- Sz. Plewik, *On subspaces of the Pixley-Roy example*. Colloq. Math. 44 (1981), no. 1, 41–46.
- T. C. Przymusiński, *Normality and paracompactness of Pixley-Roy hyperspaces*. Fund. Math. 113 (1981), no. 3, 201–219.
- R. Telgársky, *Total paracompactness and paracompact dispersed spaces*. Bull. Acad. Polon. Sci. Math. Astronom. Phys. 16 (1968), 567–572.

THANK YOU FOR ATTENTION