Wadge Hierarchy
on Second Countable Spaces

Reduction via relatively continuous relations

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Classify Definable subsets of topological spaces

\( X \) a 2\(^{nd} \) countable \( T_0 \) topological space:
- A countable basis of open sets,
- Two points which have same neighbourhoods are equal.

Borel sets are naturally classified according to their definition

\[
\Sigma_1^0(X) = \{ O \subseteq X \mid X \text{ is open} \},
\]
\[
\Sigma_2^0(X) = \left\{ \bigcup_{i \in \omega} B_i \ \bigg| \ B_i \text{ is a Boolean combination of open sets} \right\},
\]
\[
\Pi_\alpha^0(X) = \{ A^c \mid A \in \Sigma_\alpha^0(X) \},
\]
\[
\Sigma_\alpha^0(X) = \left\{ \bigcap_{i \in \omega} P_i \ \bigg| \ P_i \in \bigcup_{\beta < \alpha} \Pi_\beta^0(X) \right\}, \quad \text{for } \alpha > 2.
\]

Borel subsets of \( X = \bigcup_{\alpha < \omega_1} \Sigma_\alpha^0(X) = \bigcup_{\alpha < \omega_1} \Pi_\alpha^0(X) \)
Wadge reducibility

Let $X$ be a topological space, $A, B \subseteq X$. $A$ is Wadge reducible to $B$, or $A \leq_W B$, if there is a continuous function $f : X \to X$ that reduces $A$ to $B$, i.e. such that $f^{-1}(B) = A$ or equivalently

$$\forall x \in X \ (x \in A \iff f(x) \in B).$$

The idea is that the continuous function $f$ reduces the membership question for $A$ to the membership question for $B$.

- The identity on $X$ is continuous, and
- continuous functions compose, so

Wadge reducibility is a quasi order on subsets of $X$. Is it useful?
Hierarchies?

On Polish 0-dimensional spaces, the relation $\leq_W$ yields a nice and useful hierarchy, by results of Wadge, Martin, Monk, Louveau, Duparc and others.

Thanks to a game theoretic formulation of the reduction.
Hierarchies?

On Polish 0-dimensional spaces, the relation $\leq_W$ yields a nice and useful hierarchy, by results of Wadge, Martin, Monk, Louveau, Duparc and others.

On non 0-dim metric spaces, and many other non metrisable spaces the relation $\leq_W$ yields no hierarchy at all, by results of Schlicht, Hertling, Ikegami, Tanaka, Grigorieff, Selivanov and others.

Thanks to a game theoretic formulation of the reduction.

\[ \cdots \]

\[ \cdots \]
Reduction by continuous functions yield a nice hierarchy of subsets of Polish 0-dimensional spaces.
To get a nice hierarchy outside the realm of Polish 0-dim spaces:

- Motto Ros, Schlicht and Selivanov have considered reducibility by reasonably discontinuous functions.

We propose to weaken the second fundamental concept at stake namely, functionality:

- We want to consider reducibility by relatively continuous relations.
Reductions

Fix sets $X$, $Y$, and subsets $A \subseteq X$, $B \subseteq Y$.

A \textit{reduction} of $A$ to $B$ is a function $f : X \rightarrow Y$ such that

$$\forall x \in X \ (x \in A \iff f(x) \in B).$$

A \textit{total relation} from $X$ to $Y$ is a relation $R \subseteq X \times Y$ with

$$\forall x \in X \ \exists y \in Y \ R(x, y),$$

in symbols $R : X \Rightarrow Y$.

\textbf{Definition}

A \textit{reduction} of $A$ to $B$ is a total relation $R : X \Rightarrow Y$ such that

$$\forall x \in X \ \forall y \in Y \ (R(x, y) \rightarrow (x \in A \iff y \in B)),$$

or equivalently

$$\forall x \in X \ (x \in A \land R(x) \subseteq B) \lor (x \notin A \land R(x) \cap B = \emptyset)$$

where $R(x) = \{y \in Y : R(x, y)\}$. 
Reductions, basic properties

Basic Properties

Let \( A \subseteq X, B \subseteq Y, C \subseteq Z, \text{ and } R : X \Rightarrow Y, T : Y \Rightarrow Z : \)

- If \( R \) reduces \( A \) to \( B \) and \( T \) reduces \( B \) to \( C \), then

\[
T \circ R = \{(x, z) : \exists y \in Y \ R(x, y) \land T(y, z)\}
\]

reduces \( A \) to \( C \).

Let \( \mathcal{R} \) be a class of total relations from \( X \) to \( X \) with

1. the identity on \( X \) belongs \( \mathcal{R} \),
2. \( \mathcal{R} \) is closed under composition.

For \( A, B \subseteq X \),

\[
A \ \text{\( \mathcal{R} \)-reducible to} \ B \quad \Longleftrightarrow \quad \exists R \in \mathcal{R} \quad R \text{ reduces } A \text{ to } B
\]

This defines a quasi-order \( \leq_{\mathcal{R}} \) on subsets of \( X \).
Reductions, basic properties

Basic Properties

Let $A \subseteq X$, $B \subseteq Y$, $R, S : X \Rightarrow Y$:

- If $R \subseteq S$ and $S$ reduces $A$ to $B$, then $R$ also reduces $A$ to $B$.

Let $\mathcal{R}$ be a class of total relations from $X$ to $X$ with

1. the identity on $X$ belongs $\mathcal{R}$,
2. $\mathcal{R}$ is closed under composition.

Let $\overline{\mathcal{R}} = \{ S : X \Rightarrow X : \exists R \in \mathcal{R} \ R \subseteq S \}$, then for any $A, B \subseteq X$,

$$A \mathcal{R}\text{-reducible to } B \iff A \overline{\mathcal{R}}\text{-reducible to } B$$

In particular,

$$A \leq_{\mathcal{W}} B \iff A \overline{\mathcal{W}}\text{-reduces to } B.$$ 

where $\mathcal{W} = \{ \text{graph}(f) : f : X \to X \text{ is continuous} \}$.
Admissible representations

Let $f, g : \subseteq \omega^\omega \to X$ be partial maps. Say $f$ continuously reduces to $g$, $f \leq_W g$, if

\[ \exists \text{ continuous } r : \text{dom } f \to \text{dom } g \quad \forall \alpha \in \text{dom } f \quad f(\alpha) = g \circ r(\alpha). \]

**Proposition (Kreitz, Weihrauch, Schröder)**

Let $X$ be 2\textsuperscript{nd} countable $T_0$. There exists a partial map $\rho : \subseteq \omega^\omega \to X$ such that

- $\rho$ is continuous (and surjective),
- ($\leq_W$-greatest) $\forall$ continuous $f : \subseteq \omega^\omega \to X$, $f \leq_W \rho$.

Such a map is called an admissible representation of $X$.

If $(V_n)_{n \in \omega}$ is a basis for $X$, then one can take $\rho : \subseteq \omega^\omega \to X$:

\[ \rho(\alpha) = x \iff \{ \alpha(k) : k \in \omega \} = \{ n : x \in V_n \}. \]
Relatively continuous functions

Let $X, Y$ be 2nd countable $T_0$ spaces. A map $f : X \to Y$ is relatively continuous if for some (hence any) admissible representations $\rho_X, \rho_Y$ there exists a continuous $F : \text{dom } \rho_X \to \text{dom } \rho_Y$ such that

$$\forall \alpha \in \text{dom } \rho_X \quad f \circ \rho_X(\alpha) = \rho_Y \circ F(\alpha)$$

**Proposition**

Let $X, Y$ be 2nd countable $T_0$. A map $f : X \to Y$ is relatively continuous iff it is continuous.

If $\rho_X$ is not injective, a continuous map $F : \text{dom } \rho_X \to \text{dom } \rho_Y$ may very well induce no function from $X$ to $Y$. We can have $\alpha \neq \beta, \rho_X(\alpha) = \rho_X(\beta)$, and $\rho_Y(F(\alpha)) \neq \rho_Y(F(\beta))$. 
Definition (Brattka, Hertling, Weihrauch)

\(X, Y\) 2nd countable \(T_0\) spaces.
A total relation \(R : X \Rightarrow Y\) is \textit{relatively continuous} if
for some (hence any) admissible representations \(\rho_X, \rho_Y\)
there exists a continuous \(F : \text{dom} \, \rho_X \to \text{dom} \, \rho_Y\) such that

\[\forall \alpha \in \text{dom} \, \rho_X \quad R(\rho_X(\alpha), \rho_Y(F(\alpha)))\]

Basic Properties

1. \textit{graphs of continuous functions are relatively continuous.}
2. \textit{relatively continuous relations compose.}
3. \textit{If \(R, S : X \Rightarrow Y\), \(R\) relatively continuous and \(R \subseteq S\),
then \(S\) is also relatively continuous.}
Reduction by relatively continuous relations

**Definition**

Let $X$ be 2nd countable $T_0$, $A, B \subseteq X$. 

A *is reducible* to $B$, $A \preceq B$, if there exists a relatively continuous relation $R : X \Rightarrow X$ that reduces $A$ to $B$.

**Basic Properties**

1. $\preceq$ is a quasi order on subsets of $X$.
2. If $A \leq_W B$, then $A \preceq B$.
3. For any admissible representation $\rho$ of $X$, $A \preceq B$ iff there exists a continuous $F : \text{dom} \, \rho \to \text{dom} \, \rho$ with

\[ \forall \alpha \in \text{dom} \, \rho \quad \left( \alpha \in \rho^{-1}(A) \iff F(\alpha) \in \rho^{-1}(B) \right). \]
the case of 0-dimensional spaces

**Theorem**

Let $X$ be a $2^{nd}$ countable $T_0$ space. The following are equivalent.

1. $X$ is 0-dimensional.
2. $X$ admits an injective admissible representation.

So in a $2^{nd}$ countable 0-dim space $X$, for $R : X \Rightarrow X$:

$R$ is relatively continuous $\iff R$ admits a continuous uniformizing function.

This is not at all the case in the real line $\mathbb{R}$, for example.

**Corollary**

Let $X$ be $2^{nd}$ countable 0-dim, $A, B \subseteq X$: $A \leq_W B \iff A \preceq B$.

That is, on $2^{nd}$ countable 0-dim spaces

Wadge reducibility $= \text{reducibility by relativ. cont. relations.}$
Borel representable spaces

Definition

A 2nd countable $T_0$ space $X$ is called \textit{Borel representable space} if there exists an admissible representation $\rho$ of $X$ whose domain is Borel (in $\mathbb{N}^\mathbb{N}$).

Borel representable spaces include

- every Borel subspace of any Polish space, i.e. every Borel subspace of $[0, 1]^{\mathbb{N}}$.
- every Borel subspace of any quasi-Polish space, i.e. every Borel subspace of $\mathcal{P}(\mathbb{N})$ with the Scott topology.

Most (all?) properties of Wadge reducibility on 0-dim Polish spaces extends to arbitrary Borel representable spaces via the reducibility by relatively continuous relations.
The nice picture regained

Analysis of Wadge reducibility on $\omega^\omega$ (Wadge, Martin, Monk) and Determinacy of Borel games (Martin) directly yield

**Theorem**

Let $X$ be Borel representable.

1. For Borel sets $A, B \subseteq X$, either $A \preceq B$ or $B \preceq A^C$ (so antichains have size at most 2).
2. $\preceq$ is well founded on Borel sets.

And it follows by results of Saint Raymond and De Brecht that

**Theorem**

Let $X$ be 2nd countable $T_0$ and $\Gamma$ be $\Sigma^0_\xi$, $\Pi^0_\xi$ or $D_\theta(\Sigma^0_\xi)$. Then if $B \in \Gamma$ and $A \preceq B$, then $A \in \Gamma$. 