Ordering functions

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General background: 0-dimensional Polish spaces, and functions

- The **Baire space** ω^ω of infinite sequences of integers.
- The **product topology** on it.
- The classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ on the α-th level of the Borel hierarchy.

Each 0-dimensional Polish space is isomorphic to a closed $A \subseteq \omega^\omega$.

- A **function** is from a closed subset of ω^ω to ω^ω.
- A function f is **Baire class** α if the inverse image by f of any open subset of ω^ω is $\mathbf{\Sigma}^0_{\alpha+1}$.
- A function f is **Borel** if it is Baire class α for some $\alpha < \omega_1$.

Objective : A quasi-order on functions

Project:

Finding a solid notion to classify definable, *i.e.* Borel, functions.

- A **quasi-order** (qo) on a set Q is a reflexive and transitive relation $\leq_Q \subseteq Q^2$.
- Denote:
  - $p <_Q q$ when $p \leq_Q q$ but $q \not\leq_Q p$,
  - ($\leq_Q$-equivalence) $p \equiv_Q q$ when $p \leq_Q q$ and $q \leq_Q p$

We are looking for a qo that both:

- has nice topological properties.
- is simple enough to convey valuable information.

Requirements : being well-behaved.

Refining the Baire hierarchy

If g reduces f and g is Baire class α then so should be f.

Being well-quasi-ordered (wqo)

- **well founded**, it admits no infinite descending chain;
- no infinite antichain

An **antichain** is a set of pairwise incomparable elements.
The Wadge qo on Borel sets is wqo, we look for one that is compatible with it.

The Wadge qo on Borel sets

$A \leq_W B$ iff $A = \sigma^{-1}(B)$ iff $\mathbf{1}_A = \mathbf{1}_B \circ \sigma$ for some continuous σ.

Requirements : refining the known hierarchies

The continuous embeddability

A **continuously embeds** in B iff $g \circ f = \mathrm{Id}_A$ holds for some continuous functions $f : A \to B$ and $g : B \to A$.

The **Borel degree function of** f is

$$\begin{aligned} d_f : \omega_1 &\longrightarrow \omega_1 \\ \alpha &\longmapsto \min\{\beta \in \omega_1 \mid f^{-1}(\mathbf{\Sigma}^0_\alpha) \subseteq \mathbf{\Sigma}^0_\beta\}. \end{aligned}$$

The Borel order on functions

$f \leq_B g$ iff $d_f(\alpha) \leq d_g(\alpha)$ for all $\alpha \in \omega_1$.

Natural candidates: Wadge's qo on functions

$\leq_W$

Wadge's qo on sets can be generalised to functions:

$f \leq_W g$ iff $f = g \circ \sigma$ for some continuous σ.

It would be perfect, except..

Problem:

The constant functions form an antichain of size $2^{\aleph_0}$!

Solution:

We have to weaken the condition.

Natural candidates: Hertling and Weihrauch's (1993)

$f \leq g$ iff $f = \tau \circ g \circ \sigma$ for some continuous σ and τ.

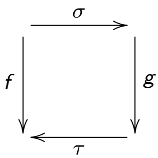
$\leq_W$

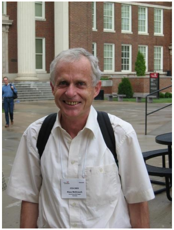
$\Downarrow$

$\leq$

$\Downarrow$

$\leq_2$





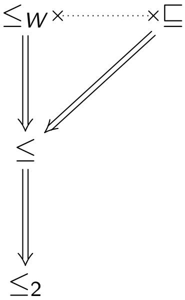
$f \leq_2 g$ iff $f(x) = \tau(x, g \circ \sigma(x))$ for continuous σ and τ.

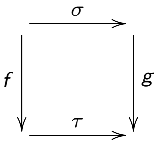
Problem:

The continuous functions are all $\leq_2$-equivalent!

Natural candidates: Solecki's (1998)

$f \sqsubseteq g$ iff $\tau \circ f = g \circ \sigma$ for some continuous **embeddings** σ and τ.

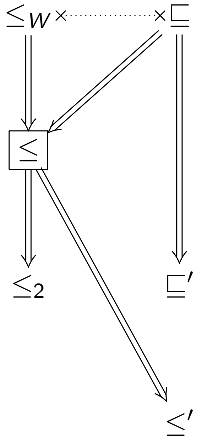




Problems:

1. $\sqsubseteq$ and $\leq_W$ are not compatible.
2. Antichains appear already among locally constant functions.

Other candidates: Can weaken the ones we have?



First try: a weakening of $\sqsubseteq$

$f \sqsubseteq' g$ iff $\tau \circ f = g \circ \sigma$ for some continuous **injection** σ and τ.

Problem:

Arbitrarily complex Borel isomorphisms are reduced by the identity.

Second try: a weakening of $\leq$

$f \leq' g$ iff $f = \tau \circ g \circ \sigma$ for some $\mathbf{\Sigma}^0_2$-measurable σ and τ.

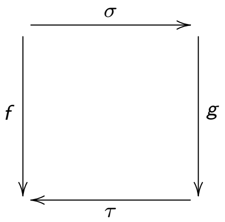
Problem:

Every continuous function is equivalent to a locally constant one.

The one we choose: Hertling and Weihrauch's strong qo

Definition

$f \leq g$ iff $f = \tau \circ g \circ \sigma$ for some continuous σ and τ.



We choose $\leq$: what are its first properties?

Properties

1. *For any* $A \subseteq \omega^\omega$, $\mathbf{1}_A \equiv \mathbf{1}_{A^c}$.
2. $A \leq_W B \Rightarrow \mathbf{1}_A \leq \mathbf{1}_B$ *and* $\mathbf{1}_A \leq \mathbf{1}_B \Rightarrow A \leq_W B$ *or* $A^c \leq_W B$.
3. $\mathrm{Id}_A \leq \mathrm{Id}_B$ *iff* A *continuously embeds into* B.
4. *If* $f \leq g$ *then* $d_f(\alpha) \leq d_g(\alpha)$ *for all* α.

This qo seems to fulfill all requirements...except one.

General Problem

Is $\leq$ a wqo on Borel functions?

This problem being much too general, we ask a more specific question.

Is $\leq$ a wqo on continuous functions?

First good news.

Proposition

If f is Borel and $\mathrm{Im}(f)$ uncountable then $\mathrm{Id}_{\omega^\omega} \leq f$.

Idea for the proof.

Find a compact $K \subseteq \mathrm{dom}(f)$ such that $f|_K$ is both continuous and injective, hence an embedding. Having a continuous inverse, an embedding reduces $\mathrm{Id}_{\omega^\omega}$, so $\mathrm{Id}_{\omega^\omega} \leq f|_K \leq f$. □

Since $\mathrm{Id}_{\omega^\omega}$ is maximal among continuous functions..

Corollary

- *If f is continuous and $\mathrm{Im}(f)$ uncountable then $f \equiv \mathrm{Id}_{\omega^\omega}$.*
- *If f is continuous then either $d_f(\alpha) = \alpha$ or $d_f(\alpha) \leq 2$.*

So we can focus on continuous functions **with countable image**.

The Cantor-Bendixson rank of a function.

Notation

C denotes the set of continuous functions with countable image.

Such a function should be locally constant "somewhere", right?

Definition

$x \in \mathrm{dom}(f)$ is f-**isolated** iff $f^{-1}(\{f(x)\})$ is neighbourhood of x iff f is locally constant on a neighborhood of x.

Proposition

If $f \in \mathrm{C}$ then the set of f-isolated points is dense open in $\mathrm{dom}(f)$.

Idea for the proof.

Otherwise the sets $f^{-1}(\{y\})$ for $y \in \mathrm{Im}(f)$ form a countable partition of $\mathrm{dom}(f)$ in nowhere dense closed sets, a contradiction with the Baire Category Theorem. □

The Cantor-Bendixson rank of a function.

Define by induction a decreasing sequence of closed sets.

Definition

- $\mathrm{CB}_0(f) = \mathrm{dom}(f)$,
- $\mathrm{CB}_{\alpha+1}(f) = \{x \in \mathrm{CB}_\alpha(f) \mid x \text{ is not } f|_{\mathrm{CB}_\alpha(f)}\text{-isolated}\}$,
- $\mathrm{CB}_\lambda(f) = \bigcap_{\alpha\in\lambda} \mathrm{CB}_\alpha(f)$ for λ limit.

If $f \in \mathrm{C}$, then for some $\alpha < \omega_1$ $\mathrm{CB}_\alpha(f) = \emptyset$.

Definition

The minimal such α is the **Cantor-Bendixson rank of** f, denoted by $\mathrm{CB}(f)$.

For a closed set F, a point $x \in F$ is isolated iff it is Id_F-isolated, so the usual Cantor-Bendixson rank of F is in fact the rank of Id_F.

The type of a function.

We have now a stratification of C.

Notation

For $\alpha < \omega_1$, denote $C_\alpha = \{f \in C \mid CB(f) = \alpha\}$.

For example, C_1 is exactly the set of locally constant functions.

Fact

For $f, g \in C_1$, $|\mathrm{Im}(f)| \leq |\mathrm{Im}(g)|$ implies $f \leq g$.

Hence C_1 is a well-order of size $\omega + 1$.

Can we generalise this fact?

If $CB(f) = \alpha + 1$, then $f|_{CB_\alpha(f)}$ is locally constant. Denote N_f the cardinal of its image. Define N_f to be 0 if $CB(f)$ is limit.

Set the **type** of $f \in C$ to be $tp(f) = (CB(f), N_f)$.

Second good news: when the domain is compact.

The type is an invariant for $\leq$.

Fact

For $f, g \in \mathsf{C}$, $f \leq g$ implies $tp(f) \leq_{lex} tp(g)$.

The converse is in general not true.
However, if $\mathsf{C}^\star = \{f \in \mathsf{C} \mid \mathrm{dom}(f) \text{ compact } \}$,

Theorem (C. 2013)

For $f, g \in \mathsf{C}^\star$, $f \leq g \Leftrightarrow tp(f) \leq_{lex} tp(g)$.

So the reduction we chose is on $\mathsf{C}^\star$ even better than a wqo!

Corollary

The class $\mathsf{C}^\star$ under $\leq$ is a well-order of size ω_1.

What about the general case?

A result on the general structure of C.

Of course the general case is more complex. However:

- If CB(f) is “much smaller” than CB(g) then $f \leq g$.
- The levels C_λ, for λ limit, are very simple.

Definition

- For n, m integers, set $n \leq^\bullet m$ iff $n = m$ or $2n < m$.
- For $\alpha \in \omega_1$ call $n_\alpha \in \omega$ such that $\alpha = \lambda_\alpha + n_\alpha$ with λ_α limit.

Now we can state precisely the result.

General Structure Theorem (C. 2013)

For $f, g \in \mathsf{C}$ such that $\alpha = \mathsf{CB}(f) \leq \mathsf{CB}(g) = \beta$,

1. *If $n_\alpha = 0$ or $\lambda_\alpha < \lambda_\beta$, then $f \leq g$.*
2. *If $\lambda_\alpha = \lambda_\beta$ and $n_\alpha <^\bullet n_\beta$ then $f \leq g$.*

Consequences.

Let F be any closed set of CB-rank a limit ordinal λ.

Corollary

$(\mathsf{C}_\lambda / \equiv) = \{\mathsf{Id}_F\}$.

Set now $\mathsf{C}' = (\sum_{\lambda \text{ limit}} \sum_{n\in\omega} \mathsf{C}_{\lambda+n}, \leq_{lex})$ to be the lexicographical quasi-order on the triples (λ, n, f) such that

- λ is limit
- integers are ordered with $\leq^\bullet$,
- $\mathsf{CB}(f) = \lambda + n$

Corollary

1. *If C' is wqo then so is C.*
2. *If C_α is wqo for all $\alpha < \omega_1$ then C is wqo.*

Consequences: embedding on closed sets.

The general case of C is still unknown, but:

Notation

- $\mathcal{A} = \{f \in \mathsf{C} \mid f \equiv \mathsf{Id}_A \text{ for some closed } A\}$.
- $\mathcal{A}_\alpha = \mathcal{A} \cap \mathsf{C}_\alpha$.

Let us apply our analysis, the induction gives

Proposition

If ($\mathcal{A}_\alpha$ wqo implies $\mathcal{A}_{\alpha+1}$ wqo) for all $\alpha < \omega_1$ then $\mathcal{A}$ is wqo.

A closed set of CB rank $\alpha + 1$ admits a decomposition into a sequence of sets of rank α, so:

Proposition

$(\mathcal{A}_\alpha)^\omega$ wqo implies $\mathcal{A}_{\alpha+1}$ wqo.

Consequences: embedding on closed sets.

Question

Does Q wqo imply Q^ω wqo?

Answer: No! However by strengthening the hypothesis on $\mathcal{A}_\alpha$..

Proposition

$\mathcal{A}_\alpha$ better-quasi-order (bqo) $\Rightarrow (\mathcal{A}_\alpha)^\omega$ bqo $\Rightarrow \mathcal{A}_{\alpha+1}$ bqo.

Finally, piecing everything together,

Theorem

Continuous embeddability is a wqo on the closed subsets of ω^ω.

Conjecture

$(\mathsf{C}, \leq)$ is a wqo.

Conjecture

The class of Borel functions, ordered by $\leq$, is a wqo.

Conjecture

Given a Borel function f, there is a closed set $F \subseteq \mathrm{dom}(f)$ such that

- $f|_F$ is $\leq$-equivalent to a Borel isomorphism,
- $\mathrm{Im}(f|_{F^c})$ is countable.

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- $f|_F$ is $\leq$-equivalent to a Borel isomorphism,
- $\mathrm{Im}(f|_{F^c})$ is countable.

Thank you!