The Filter Dichotomy

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Definition (Filter Dichotomy, **FD**)

The Filter Dichotomy is the principle which says: for any free filter $\mathcal{F}$ on ω there is a finite-to-one map $\phi\colon \omega \to \omega$ such that either

1. $\phi(\mathcal{F}) = \mathcal{C}of$, where $\mathcal{C}of$ denotes the cofinite filter, or
2. $\phi(\mathcal{F}) = \mathcal{U}$ where U is a free ultrafilter on ω.

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This is secretly topology - if we stand on our heads (and use Stone Duality) we see any discussion of free filters is really a discussion of closed subsets of $\omega^* = \beta\omega \setminus \omega$, the Stone-Čech remainder of ω. Some lazy terminology: if there is a finite-to-one map sending a filter $\mathcal{F}$ to a filter $\mathcal{G}$, we will say more concisely that $\mathcal{F}$ is almost $\mathcal{G}$.

An easy observation...

If $\mathcal{F}$ is a free filter on ω and $\chi(F) < \mathfrak{u}$ then $\mathcal{F}$ cannot be almost an ultrafilter. Hence under **FD** $\mathcal{F}$ is almost $\mathcal{C}of$.

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Consistency of **FD** is known, but in all models so far constructed, $\mathfrak{u} = \aleph_1$. It is easy to see however, in **ZFC** that if a free filter is countably based, then it is almost $\mathcal{C}of$ (take a pseudointersection and then it is clear how to define the map).

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Question

Is it consistent that both $\mathfrak{u} > \aleph_1$ and the Filter Dichotomy holds?

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Theorem

Assume **MA***, or more generally* $\mathfrak{p} = \mathfrak{c}$*. Then if* $\mathcal{F}$ *is a free filter on* ω *with character less than* $\mathfrak{c}$ *then* $\mathcal{F}$ *is almost* $\mathcal{C}of$*.*

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This is all well and good, but

Theorem

Under $\mathfrak{p} = \mathfrak{c}$*, the Filter Dichotomy fails.*

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Consistency so far...

The Filter Dichotomy is consistent, but only so far with $\mathfrak{u} = \aleph_1$. **MA** implies the failure of the Filter Dichotomy.

Main Question

Is it consistent that $\mathfrak{u} < \mathfrak{g}$ and $\mathfrak{u} > \aleph_1$?

So what's the problem?

Limited machinery

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Non-trivial and weird relationships

Weird relationships that complicate things...

Theorem (Shelah)

$\mathfrak{g} \leq \mathfrak{b}^+$.

Not only is this last result of Shelah seemingly mad, it has the awkward consequence that if $\mathfrak{u} < \mathfrak{g}$ then $\mathfrak{g} = \mathfrak{u}^+$! This interacts badly with iterated forcing machinery.

The easiest things just won't work

The following is not so difficult to show

Theorem

In any forcing extension by a finite support iteration of c.c.c. forcings, $\mathfrak{g} \leq \mathfrak{u}$.

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What about the (slightly weaker) Filter Dichotomy? Well one similarly shows that in a finite support iteration of c.c.c. forcings either

- The Filter Dichotomy fails, or
- "Nothing is gained from iterating the forcings". That is to say, the Filter Dichotomy is reflected in many initial stages, and we would have essentially had to cook up a one step forcing.

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Annoying since iteration is such a natural way of thinking about these problems.

A vague plan

Because of Shelah's result...

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$\mathfrak{g} \leq \mathfrak{b}^+$. *In particular if* $\mathfrak{u} < \mathfrak{g}$ *then* $\mathfrak{g} = \mathfrak{u}^+$.

...we would like to build a forcing extension in which $\mathfrak{u} = \kappa$, and $\mathfrak{g} = \kappa^+$. Iteration in the normal sense doesn't seem quite good enough, and we want to simultaneously lift $\mathfrak{u}$ to κ and $\mathfrak{g}$ to κ^+. How might we achieve these two tasks of "different complexity" at the same time?

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Idea

Build a forcing along a gap-1 morass at κ.

So far

The arguments for normal (linear) iterations seem to break down. Using forcings along a gap-1 morass at κ (in the sense of Irrgang) one obtains a model in which

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- $\mathfrak{u} \geq \kappa$, but
- $\mathfrak{g} \leq \kappa^+$.

This class of forcings might just work then. Work is ongoing to try and find an appropriate forcing...