Abstract Well and Better Quasi-Orders

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- We can think of the equivalent definition, that there is no function $f : \mathbb{N} \to Q$ such that $x < y$ implies $f(x) \not\leq f(y)$.

Fronts

A front $\mathcal{F}$ on $A \subseteq \mathbb{N}$ is a set of finite sequences of natural numbers with the following properties:

- $\mathcal{F}$ contains an initial segment of every infinite increasing sequence of natural numbers in A.
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We can define a ranking on fronts which we call the depth. The front consisting of length 1 sequences will have depth 1.

Structured Fronts

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- We define the following relation on a front:
  For $a, b \in \mathcal{F}$, $a \lhd b$ iff there is an infinite sequence X such that $a \sqsubset X$ and $b \sqsubset X^+$.

Better Quasi-Orders

- So we have another equivalent definition of WQO:
  there is no $f : \mathcal{F} \to Q$ for $\mathcal{F}$ a front of depth 1, such that $a \lhd b$ implies $f(a) \not\leq f(b)$.

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- Such an f is called bad.

Ramsey Spaces

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- Abstract Nash-Williams Theorem:
  For every front $\mathcal{F}$ on $A \in \mathcal{R}$ and every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, there is a $B \leq A$ such that $\mathcal{F}|B \subseteq \mathcal{F}_0$ or $\mathcal{F}|B \subseteq \mathcal{F}_1$.

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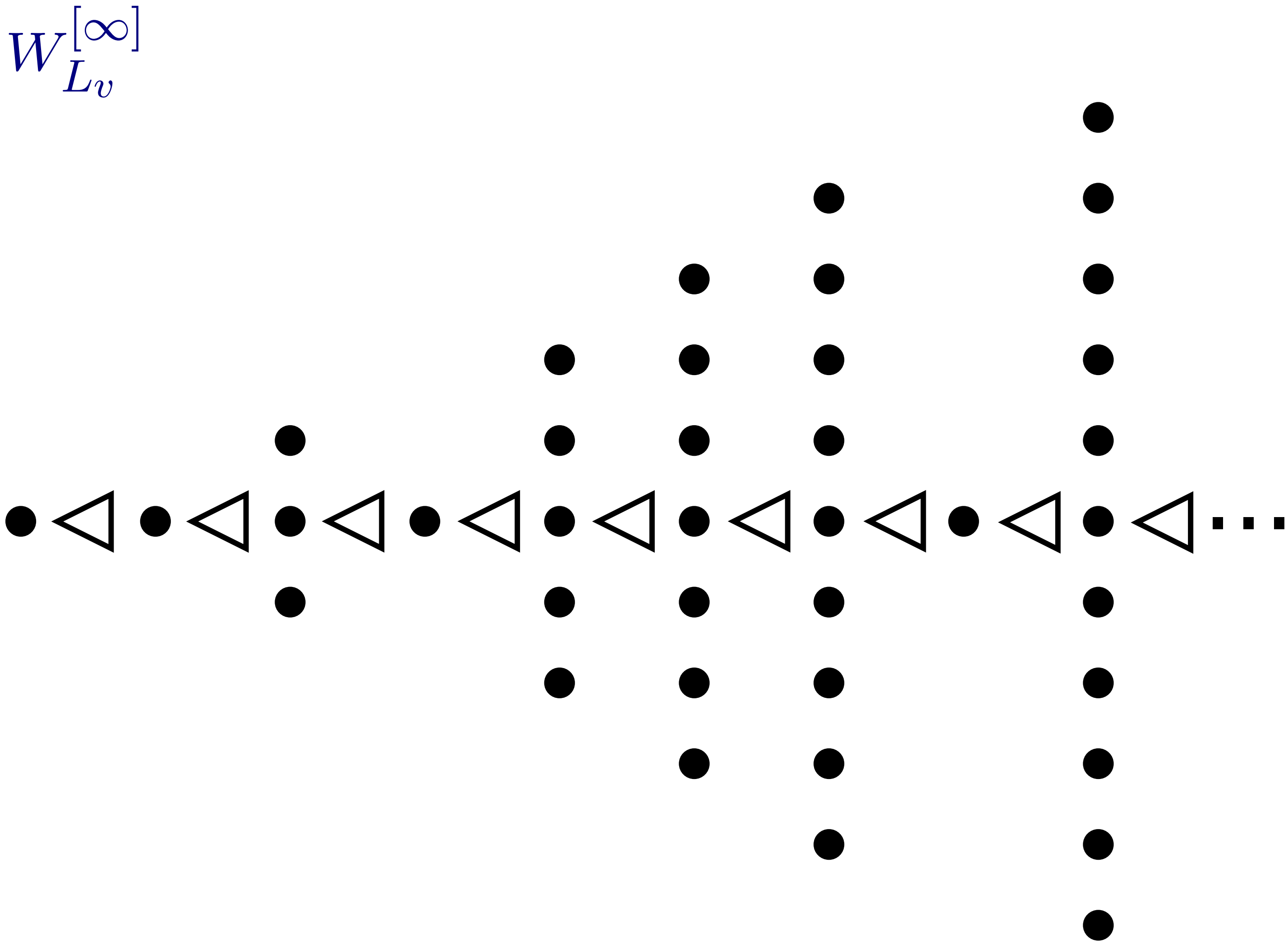
- Here the fronts are from $\mathcal{R}$ instead of $\mathbb{N}^{[\infty]}$.

$\mathcal{R}$-WQO and $\mathcal{R}$-BQO

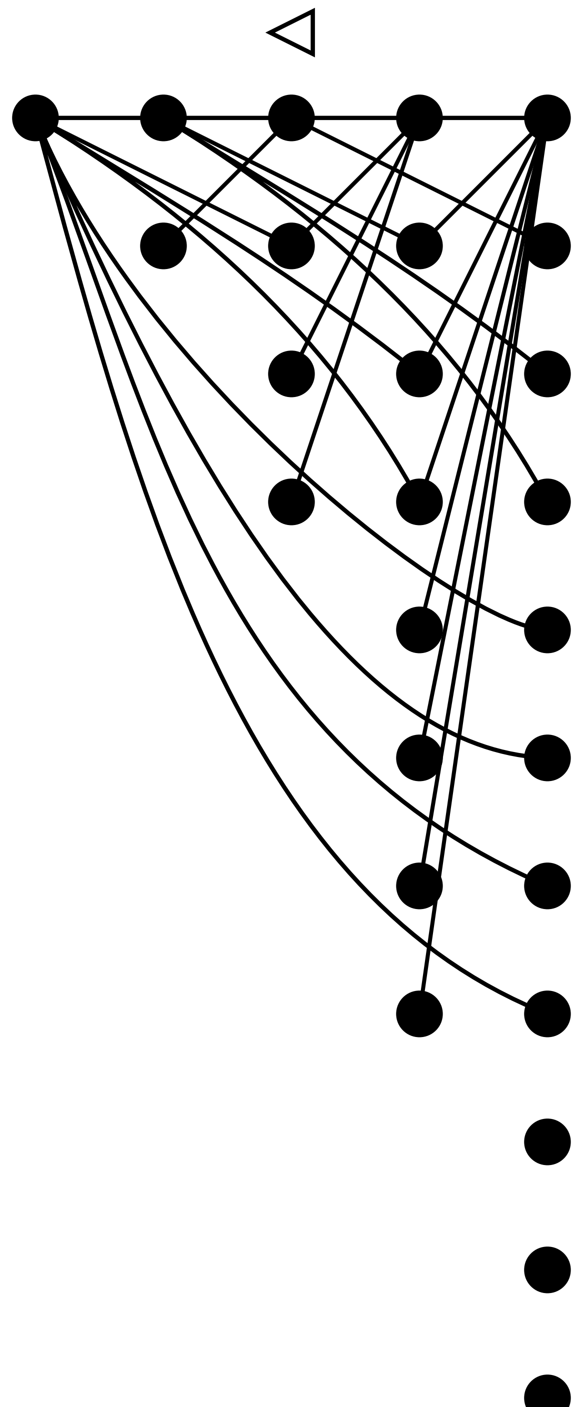
Theorem:

For any topological Ramsey space $\mathcal{R}$ that has a countable front, and any quasi-order Q, "Q is $\mathcal{R}$-WQO" is equivalent to one of the following:

- Q is any quasi-order,
- Q has no infinite antichains,
- Q has no infinite descending sequences.


$W_{L_v}^{[\infty]}$

$\mathrm{FIN}_1^{[\infty]}$



$(\mathcal{R})$-WQO and $(\mathcal{R})$-BQO

- For $a, b \in \mathcal{F}$ say $a \triangledown b$ if $r_1(a) \neq r_1(b)$ and $r_1(a) \not\triangleleft r_1(b) \not\triangleleft r_1(a)$.

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- Q is $(\mathcal{R})$-BQO iff there is no
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- Q is $(\mathcal{R})$-WQO iff there is no such f from a front of depth 1.

$(\mathcal{R})$-WQO and $(\mathcal{R})$-BQO

For a special type of $\mathcal{R}$ and by choosing a sufficiently strong $\sim$, useful techniques from BQO theory still work.

$(\mathcal{R})$-WQO and $(\mathcal{R})$-BQO

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- Minimal bad Q-array lemma.
- Q is $(\mathcal{R})$-BQO implies $\tilde{Q}$ is $(\mathcal{R})$-BQO.
- Bad functions from "Borel measurable bad functions" (Simpson's definition).

$$(\mathbb{N}^{[\infty]})\text{-WQO} \rightarrow (W_{L_v}^{[\infty]})\text{-WQO} \rightarrow (\mathsf{FIN}_1^{[\infty]})\text{-WQO} \leftrightarrow (\mathsf{FIN}_k^{[\infty]})\text{-WQO}$$

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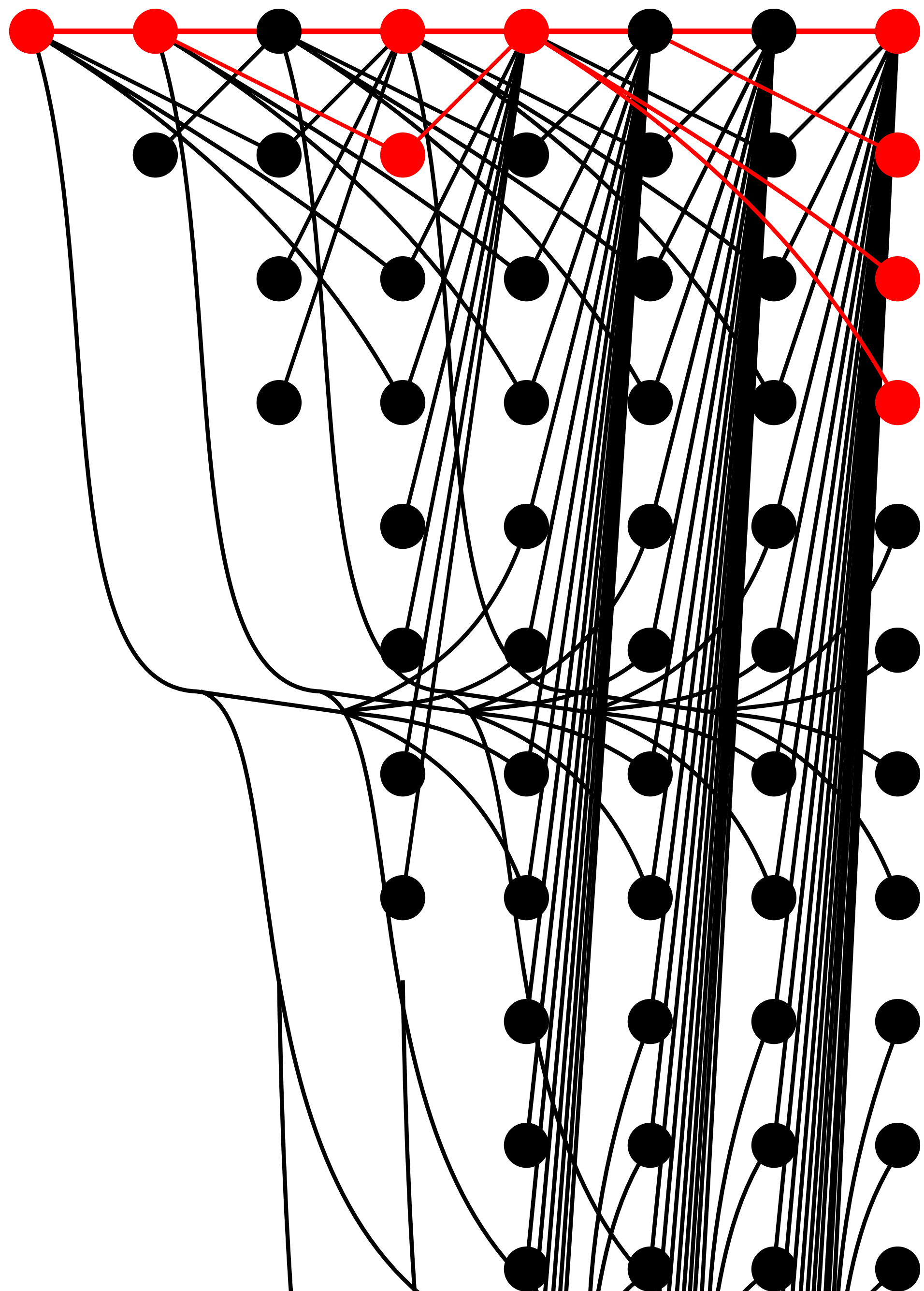
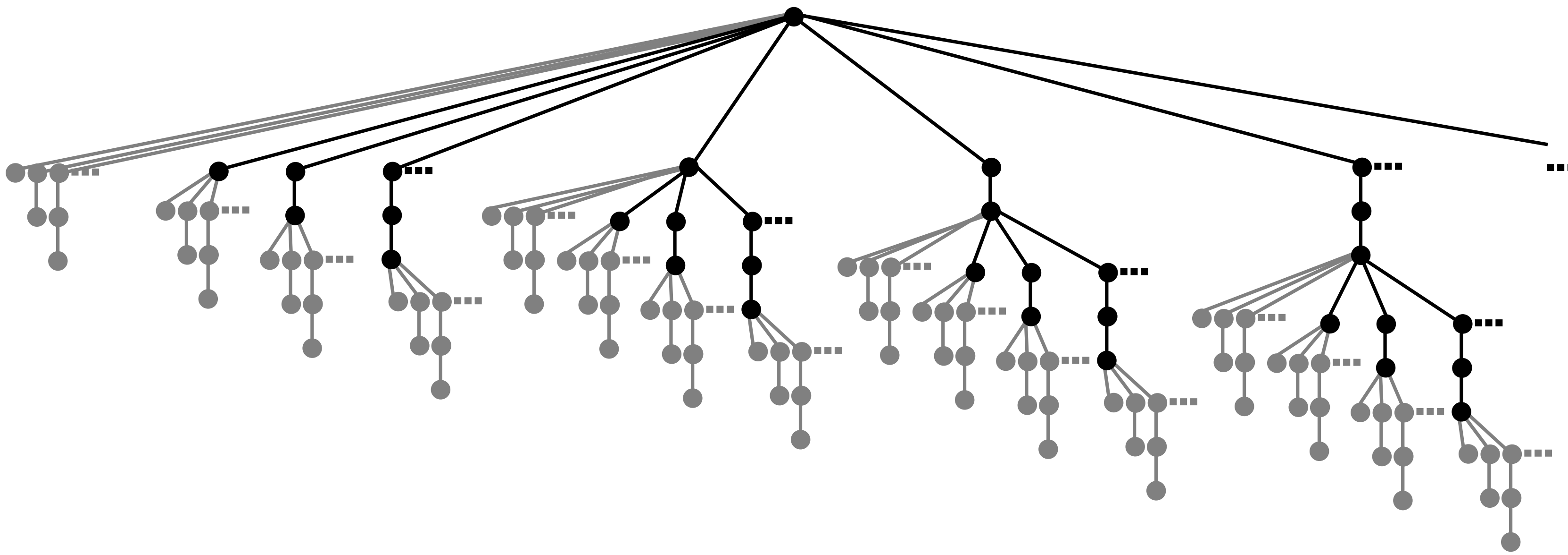


Non-Persistent Trees

Let $\mathbb{T}$ be the set of non-persistent trees of size $\aleph_1$, with no uncountable branches.



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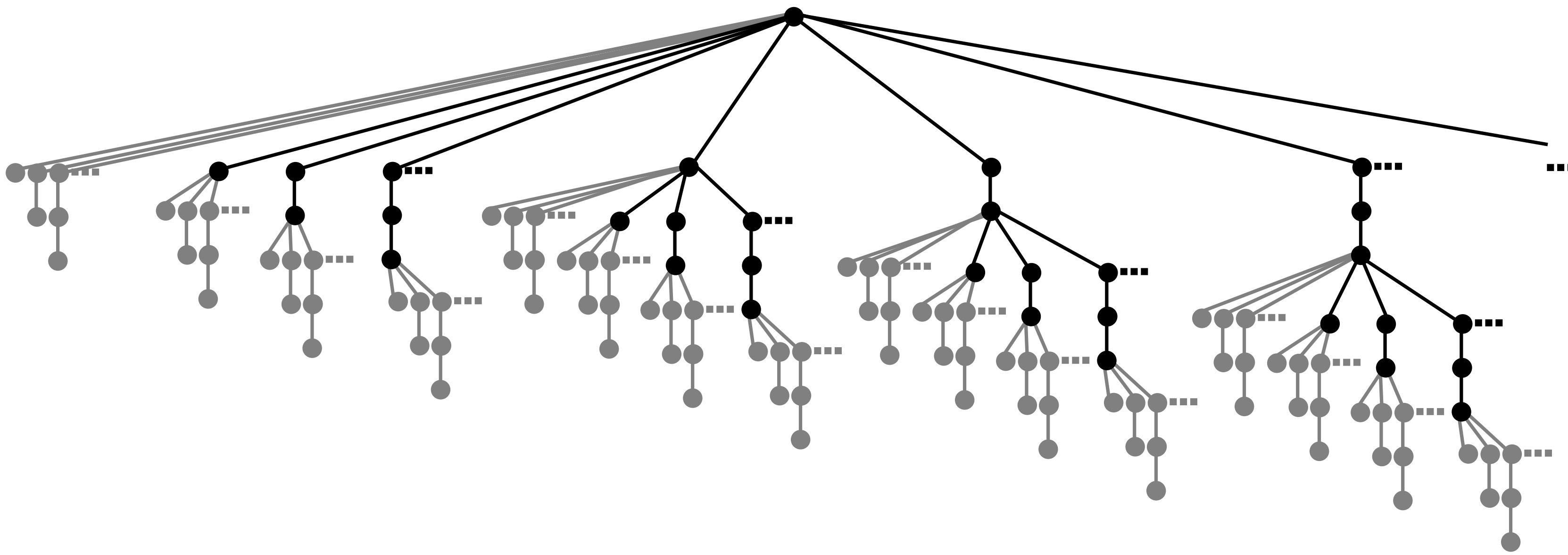

For $S, T \in \mathbb{T}$ define $S \leq T$ iff there is an $f : S \to T$ such that $a <_S b \longrightarrow f(a) <_T f(b)$.

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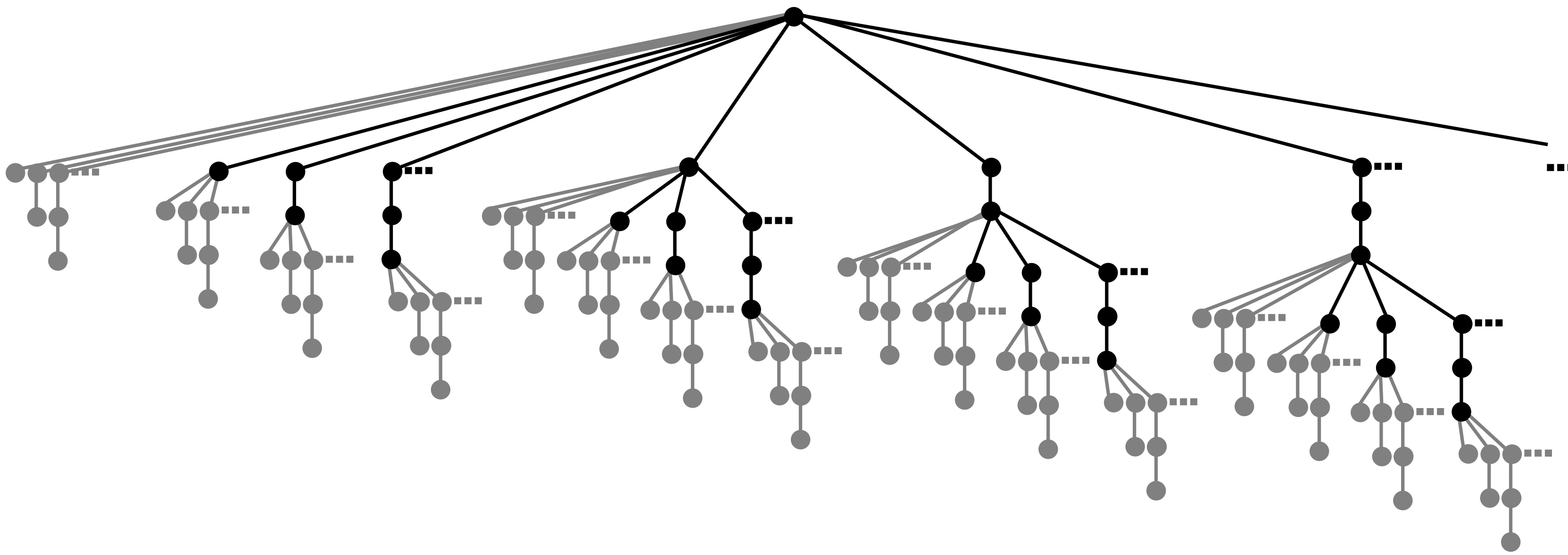

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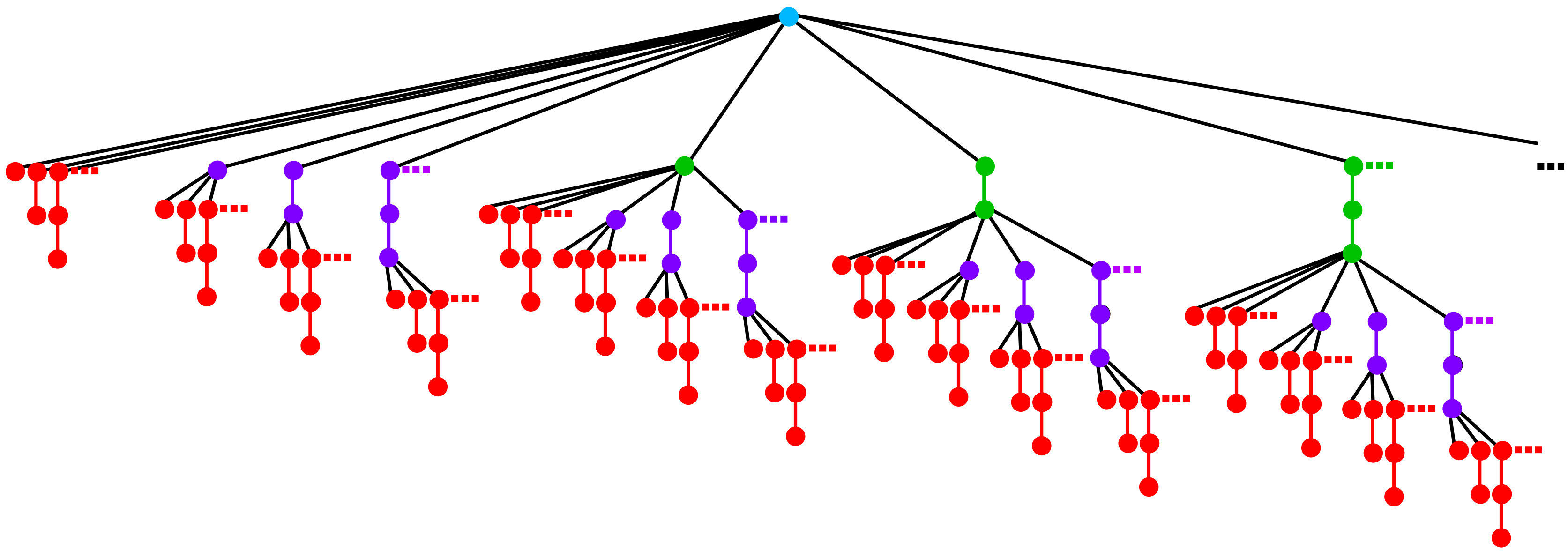
Todorčević and Väänänen proved that this order has antichains of size $2^{\aleph_1}$.

Non-Persistent Trees

Theorem:

$(\mathbb{T}, \leq, \sim)$ is $(W_{L_v}^{[\infty]})$-BQO.



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