On tightness in the space of measures on Boolean algebras and compact spaces

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Winter School 2014, Hejnice

Joint work with Grzegorz Plebanek

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$[0, \omega_1]$ and 2^{ω_1} do not have countable tightness.

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Equivalently, μ has a countable type if $L_1(\mu)$ is separable.

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$$\lambda(c_\xi \triangle c_\eta) = \frac{1}{2} \text{ whenever } \xi \neq \eta$$

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Fremlin '97

Assume $\mathrm{MA}(\omega_1) + \neg\mathrm{CH}$. Let $\mathcal{A}$ be a Boolean algebra. Let there exist $\mu \in P(\mathcal{A})$ with uncountable type.

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Pol's open question from 80-ties

If $P(K)$ has countable tightness, does $P(K \times K)$ have it also?

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Every such ν_ξ can be extended to $\overline{\nu_\xi} \in P(K \times K)$.

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Let $\mathcal{A}_2 := alg(\mathcal{A}_1 \cup \{B_{\eta_2}\})$,

Proof – Step 2, cont.

Proof

Fix $\xi < \omega_1$.

Take $A_1, \ldots, A_n \in \mathcal{C}_\xi$ and $B_{\eta_1}, \ldots, B_{\eta_m}$ for some $\xi \leq \eta_1 < \ldots < \eta_m$.

$\mathcal{A}_0 := alg(\{A_1, \ldots, A_m\})$, $\mathcal{A}_1 := alg(\mathcal{A}_0 \cup \{B_{\eta_1}\})$.

Let $\nu_0 := \mu \otimes \mu|_{\mathcal{A}_0 \times \mathcal{A}_0}$. We will extend ν_0 to $\nu_1 \in P(\mathcal{A}_1 \times \mathcal{A}_1)$:

Let $T_1, \ldots, T_k$ be all the atoms of $\mathcal{A}_0$. Put for all $i, j \leq k$:

$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B_{\eta_1})) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1}^c \times B_{\eta_1}^c)) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1}^c \times B_{\eta_1})) = 0$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B_{\eta_1}^c)) = 0$$

Let $\mathcal{A}_2 := alg(\mathcal{A}_1 \cup \{B_{\eta_2}\})$, extend ν_1 to $\nu_2 \in P(\mathcal{A}_2 \times \mathcal{A}_2)$

Proof – Step 2, cont.

Proof

Fix $\xi < \omega_1$.

Take $A_1, \ldots, A_n \in \mathcal{C}_\xi$ and $B_{\eta_1}, \ldots, B_{\eta_m}$ for some $\xi \leq \eta_1 < \ldots < \eta_m$.

$\mathcal{A}_0 := alg(\{A_1, \ldots, A_m\})$, $\mathcal{A}_1 := alg(\mathcal{A}_0 \cup \{B_{\eta_1}\})$.

Let $\nu_0 := \mu \otimes \mu|_{\mathcal{A}_0 \times \mathcal{A}_0}$. We will extend ν_0 to $\nu_1 \in P(\mathcal{A}_1 \times \mathcal{A}_1)$:

Let $T_1, \ldots, T_k$ be all the atoms of $\mathcal{A}_0$. Put for all $i, j \leq k$:

$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B_{\eta_1})) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B^c_{\eta_1} \times B^c_{\eta_1})) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B^c_{\eta_1} \times B_{\eta_1})) = 0$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B^c_{\eta_1})) = 0$$

Let $\mathcal{A}_2 := alg(\mathcal{A}_1 \cup \{B_{\eta_2}\})$, extend ν_1 to $\nu_2 \in P(\mathcal{A}_2 \times \mathcal{A}_2)$ and so on... to $\nu_m \in P(\mathcal{A}_m \times \mathcal{A}_m)$.

Proof – Step 2, cont.

Proof
Fix $\xi < \omega_1$.
Take $A_1, \ldots, A_n \in \mathcal{C}_\xi$ and $B_{\eta_1}, \ldots, B_{\eta_m}$ for some $\xi \leq \eta_1 < \ldots < \eta_m$.
$\mathcal{A}_0 := alg(\{A_1, \ldots, A_m\})$, $\mathcal{A}_1 := alg(\mathcal{A}_0 \cup \{B_{\eta_1}\})$.
Let $\nu_0 := \mu \otimes \mu|_{\mathcal{A}_0 \times \mathcal{A}_0}$. We will extend ν_0 to $\nu_1 \in P(\mathcal{A}_1 \times \mathcal{A}_1)$:

Let $T_1, \ldots, T_k$ be all the atoms of $\mathcal{A}_0$. Put for all $i, j \leq k$:

$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B_{\eta_1})) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1}^c \times B_{\eta_1}^c)) = \frac{1}{2}\nu_0(T_i \times T_j)$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1}^c \times B_{\eta_1})) = 0$$
$$\nu_1((T_i \times T_j) \cap (B_{\eta_1} \times B_{\eta_1}^c)) = 0$$

Let $\mathcal{A}_2 := alg(\mathcal{A}_1 \cup \{B_{\eta_2}\})$, extend ν_1 to $\nu_2 \in P(\mathcal{A}_2 \times \mathcal{A}_2)$ and so on... to $\nu_m \in P(\mathcal{A}_m \times \mathcal{A}_m)$. ■

Proof – Step 3

If $\nu \in \bigcap_{\xi<\omega_1} \overline{\{\overline{\nu_\eta} : \eta \geq \xi\}}$, then $\nu \notin \overline{\{\overline{\nu_\eta} : \eta \in I\}}$ for every $I \in [\omega_1]^\omega$.

Proof – Step 3

If $\nu \in \bigcap_{\xi<\omega_1} \overline{\{\overline{\nu_\eta}:\ \eta \geq \xi\}}$, then $\nu \notin \overline{\{\overline{\nu_\eta}:\ \eta \in I\}}$ for every $I \in [\omega_1]^\omega$.

The *Proof* just relies on some computations exploiting regularity of μ. ■

Proof – Step 3

If $\nu \in \bigcap_{\xi<\omega_1} \overline{\{\overline{\nu_\eta}:\ \eta \geq \xi\}}$, then $\nu \notin \overline{\{\overline{\nu_\eta}:\ \eta \in I\}}$ for every $I \in [\omega_1]^\omega$.

The *Proof* just relies on some computations exploiting regularity of μ. ■

This all gives a contradiction and the end of the proof.

The end

Thank you for your attention.