On PID and biorthogonal systems

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What is the relation between the "size" of the space and the largest "size" of a biorthogonal system?

Examples

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- Separable Banach spaces with a Schauder basis.
- Separable Banach spaces (Markushevich).

Nonseparable Banach spaces - examples

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If K is a compact space containing a nonseparable space, then $C(K)$ has an uncountable biorthogonal system.

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- B., Koszmider, 2011: consistently, there exists an example of weight ω_2.

Nonseparable Banach spaces - nonexistence results

Theorem (Todorcevic, 2006)

Under PID $+ \mathfrak{p} > \omega_1$*, every nonseparable Banach space has an uncountable biorthogonal system.*

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Are the following equivalent under the PID?

- $\mathfrak{b} = \omega_1$.
- There exists a nonseparable Banach space with no uncountable biorthogonal systems.

Nonseparable Banach spaces - nonexistence results

Theorem (B.,Todorcevic)

Under PID $+ \mathfrak{b} > \omega_1$, *every nonseparable Banach space with weak***-sequentially separable dual ball has uncountable* ε*-biorthogonal systems for every* $0 < \varepsilon < 1$.

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Under PID $+ \mathfrak{b} > \omega_1$*, every nonseparable Banach space with weak*-sequentially separable dual ball has uncountable* ε*-biorthogonal systems for every* $0 < \varepsilon < 1$.

Corollary

Under PID, the following are equivalent:

- $\mathfrak{b} = \omega_1$.
- *There exists a nonseparable Asplund space with no uncountable almost biorthogonal systems.*

Sketch of the proof

P-ideal dichotomy: If $\mathcal{I} \subset [\omega_1]^\omega$ is a P-ideal, then

- *either $\exists$ an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^\omega \subseteq \mathcal{I}$;*
- *or $\exists$ a partition $\omega_1 = \bigcup_{n\in\omega} S_n$ such that $[S_n]^\omega \cap \mathcal{I} = \emptyset$.*

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- *or* $\exists$ *a partition* $\omega_1 = \bigcup_{n\in\omega} S_n$ *such that* $[S_n]^\omega \cap \mathcal{I} = \emptyset$.

Given $\mathcal{F} \subseteq [\omega_1]^\omega$ *such that* $|\mathcal{F}| < \mathfrak{b}$, *then*

$$\mathcal{I} = \{A \in [\omega_1]^\omega : (\forall F \in \mathcal{F}) \quad |F \cap A| < \omega\}$$

is a P-ideal.

Suppose $(h_\alpha)_{\alpha\in\omega_1} \subseteq X^*$ is a (normalized) family such that

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Finally we construct an almost biorthogonal system.