On PID and biorthogonal systems

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A biorthogonal system in a Banach space $X$ is a family $(x_\alpha, f_\alpha)_{\alpha \in \kappa}$ in $X \times X^*$ such that $f_\alpha(x_\beta) = \delta_{\alpha \beta}$. 
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What is the relation between the “size” of the space and the largest “size” of a biorthogonal system?
Examples

- Finite dimensional spaces (linear algebra).
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- Separable Banach spaces with a Schauder basis.
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- Separable Banach spaces with a Schauder basis.
- Separable Banach spaces (Markushevich).
Nonseparable Banach spaces - examples

Theorem (Todorcevic, 2006)

*If K is a compact space containing a nonseparable space, then $C(K)$ has an uncountable biorthogonal system.*
Nonseparable Banach spaces - examples

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**Theorem (folklore, Negrepontis, 1984)**

*If K is a compact scattered space and \( K^n \) is hereditarily separable, then \( C(K) \) has no uncountable biorthogonal systems.*
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- Kunen, 80's: under CH, there exists a nonmetrizable example.
- Todorcevic, 80's: under $\mathfrak{b} = \omega_1$, there exists a nonmetrizable example.
- B., Koszmider, 2011: consistently, there exists an example of weight $\omega_2$. 
Theorem (Todorcevic, 2006)

Under PID + \( p > \omega_1 \), every nonseparable Banach space has an uncountable biorthogonal system.
Nonseparable Banach spaces - nonexistence results

Theorem (Todorcevic, 2006)

*Under PID + \( p > \omega_1 \), every nonseparable Banach space has an uncountable biorthogonal system.*

Are the following equivalent under the PID?

- \( b = \omega_1 \).
- There exists a nonseparable Banach space with no uncountable biorthogonal systems.
Theorem (B., Todorcevic)

Under PID + $\mathfrak{b} > \omega_1$, every nonseparable Banach space with weak*-sequentially separable dual ball has uncountable $\varepsilon$-biorthogonal systems for every $0 < \varepsilon < 1$. 

Corollary

Under PID, the following are equivalent:

- $\mathfrak{b} = \omega_1$
- There exists a nonseparable Asplund space with no uncountable almost biorthogonal systems.
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Under PID, the following are equivalent:

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- There exists a nonseparable Asplund space with no uncountable almost biorthogonal systems.
Sketch of the proof

P-ideal dichotomy: If $\mathcal{I} \subset [\omega_1]^{\omega}$ is a P-ideal, then

- either $\exists$ an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^{\omega} \subseteq \mathcal{I}$;
- or $\exists$ a partition $\omega_1 = \bigcup_{n \in \omega} S_n$ such that $[S_n]^{\omega} \cap \mathcal{I} = \emptyset$.
Sketch of the proof

**P-ideal dichotomy:** If $I \subseteq [\omega_1]^\omega$ is a P-ideal, then

- either $\exists$ an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^\omega \subseteq I$;
- or $\exists$ a partition $\omega_1 = \bigcup_{n \in \omega} S_n$ such that $[S_n]^\omega \cap I = \emptyset$.

Given $\mathcal{F} \subseteq [\omega_1]^\omega$ such that $|\mathcal{F}| < b$, then

$$I = \{ A \in [\omega_1]^\omega : (\forall F \in \mathcal{F}) \ |F \cap A| < \omega \}$$

is a P-ideal.
Suppose \((h_\alpha)_{\alpha \in \omega_1} \subseteq X^*\) is a (normalized) family such that
\[
\forall x \in X \quad (h_\alpha(x))_{\alpha \in \omega_1} \in \ell_\infty(\omega_1)
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Suppose \((h_\alpha)_{\alpha \in \omega_1} \subseteq X^*\) is a (normalized) family such that
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\forall x \in X \quad (h_\alpha(x))_{\alpha \in \omega_1} \in \ell_\infty(\omega_1) \quad \text{(equivalently, \(\{\alpha : h_\alpha(x) \neq 0\}\) is countable)}
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and \(D\) is a dense \(\mathbb{Q}\)-linear subspace of \(X\).
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and \( D \) is a dense \( \mathbb{Q} \)-linear subspace of \( X \).

Then we extract a family \( (f_\alpha)_{\alpha \in \omega_1} \) such that

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Next we extract an uncountable subfamily \((f_\alpha)_{\alpha \in \Gamma}\) such that
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Finally we construct an almost biorthogonal system.