Universal groups, Fraïssé classes of groups and group structures on the Urysohn space

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Introduction-Fraïssé theory and the Urysohn space

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- has the joint embedding property, i.e. if $A_1, A_2 \in \mathcal{K}$, then there is some $B \in \mathcal{K}$ and embeddings $\iota_i : A_i \hookrightarrow B$, for $i \in \{1, 2\}$

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- has the amalgamation property, i.e. for every $A_0, A_1, A_2 \in \mathcal{K}$ such that there are embeddings $\iota_i : A_0 \hookrightarrow A_i$, for $i \in \{1, 2\}$, then there is $A_3 \in \mathcal{K}$ and embedding $\rho_i : A_i \hookrightarrow A_3$, for $i \in \{1, 2\}$, such that $\rho_2 \circ \iota_2 = \rho_1 \circ \iota_1$.

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Then we call $\mathcal{K}$ a Fraïssé class.

The Fraïssé theorem

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Equivalently, K has the "finite extension property": let $A', B' \in \mathcal{K}$ such that $A' \subseteq B'$. Let $A \subseteq K$ be isomorphic to A'. Then there is $A \subseteq B \subseteq K$ and the isomorphism between A and A' extends to an isomorphism between B and B'.

Example - the Urysohn space

Let $\mathcal{U}$ be the class of all finite rational metric spaces. It can be proved it is a Fraïssé class so it has a Fraïssé limit denoted $\mathbb{U}_{\mathbb{Q}}$ and called the rational Urysohn space.

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Let $\mathbb{U}$ be the completion of $\mathbb{U}_{\mathbb{Q}}$. It contains every separable metric space as a subspace and every partial isometry between two finite subspaces extends to an autoisometry of $\mathbb{U}$.

Universal abelian Polish group

Theorem (Shkarin, 1999)

There exists a universal abelian Polish group G. That is, for every separable Hausdorff abelian group H there exists a subgroup $H' \leq G$ such that H and H' are topologically isomorphic.

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Thus there is a (Fraíssé) limit $G_{\mathbb{Q}}$. The group operations extend to the metric completion of $G_{\mathbb{Q}}$ which is the desired group G.

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Question (Shkarin, 1999)

Does there exist a separable abelian metric group which is metrically universal?

Abelian group structure on the Urysohn space

Theorem (P. Cameron, A. Vershik, 2006)

There is an isometry ϕ of $\mathbb{U}$ such that the ϕ-orbit of any point $x \in \mathbb{U}$, $\{\phi^n(x) : n \in \mathbb{Z}\}$, is dense in $\mathbb{U}$.

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Corollary (C.,V., 2006)

There is a monothetic (in particular abelian) group structure on the Urysohn space.

Abelian group structure on the Urysohn space

Theorem (P. Niemiec, 2009)

There exists a structure of a Boolean abelian group on the Urysohn space. Moreover, this is group is metrically universal for the class of separable Boolean abelian groups equipped with invariant metric.

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How to construct it: Take the class of all finite Boolean abelian groups with invariant rational metric.
It is again a Fraïssé class, and the completion of the Fraïssé limit is the desired group.

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G_n is isometric to the Urysohn space iff $n = 2$ or $n = \infty$.

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Question (Vershik, Niemiec)

Is Shkarin/Niemiec's group the same as Cameron-Vershik group?

Metrically universal abelian group

Theorem

There exists a separable abelian group $\mathbb{G}$ equipped with invariant metric which is metrically universal, i.e. for any separable abelian group H equipped with invariant metric there exists a subgroup $H' \leq \mathbb{G}$ such that H and H' are isometrically isometric. Moreover, $\mathbb{G}$ is isometric to the Urysohn space and is isometrically isomorphic neither to the Shkarin/Niemiec group nor to Cameron-Vershik group.

Metrically universal abelian group

Definition

Let (G, d) be a group with a two-sided invariant metric. Let us say that the metric d is generated by (values on) a set $A \subseteq G^2$ if for every $g, h \in G$ we have

$$d(g,h) = \inf\{d(a_1,b_1) + \ldots + d(a_n,b_n) : n \geq 1, g = a_1 \cdot \ldots \cdot a_n,$$

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Fact

If the metric d on G is generated by a finite set on which it attains rational values, then d is rational.

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Theorem

$\mathcal{G}$ is a (generalized) Fraïssé class.

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Fact

Let $F \leq \mathbb{G}_{\mathbb{Q}}$ be a finitely generated subgroup of $\mathbb{G}_{\mathbb{Q}}$ that is isometrically isomorphic to a direct summand of $\mathbb{G}_{\mathbb{Q}}$. Let $H \in \mathcal{G}$ a consider $F \oplus H$ with some finitely generated rational metric that extends those on F, H respectively. Then there exists an isometric isomorphism between $F \oplus H$ and a direct summand of $\mathbb{G}_{\mathbb{Q}}$ that is the identity on F.

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- For every $a, b \in \mathbb{G}_{\mathbb{Q}}$ we have $d(a, b) = d(-a, -b)$.
- For every $g_1, g_2, h_1, h_2 \in \mathbb{G}_{\mathbb{Q}}$ we have $d(g_1+h_1, g_2+h_2) = d(g_1-g_2, h_2-h_1) \leq d(g_1, g_2)+d(h_1, h_2)$.

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We now sketch the idea why $\mathbb{G}$ is metrically universal and why it is isometric to the Urysohn space.

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Let X be a metric space. Recall that a function $f : X \to \mathbb{R}^+$ is called Katětov if for every $x, y \in X$ we have

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Think about f as a prescription of distances from some new point to the points of X.

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Proposition

Let G be an abelian group with invariant metric. Let $A \subseteq G$ be a finite subset and $f : A \to \mathbb{R}^+$ a Katětov function. Then there exists an extension of the metric on G to $G \oplus \mathbb{Z}$ such that the new generator realizes f.

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Moreover, if the metric on G was rational and finitely generated, f attains only rational values, then the extended metric can be also made rational and finitely generated.

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Corollary

$\mathbb{G}_{\mathbb{Q}}$ is isometric to the rational Urysohn space. It follows that $\mathbb{G}$ is isometric to the Urysohn space.

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- It is a countable rational metric space.
- Every rational Katětov function with finite domain is realized.

Using the previous proposition one can check that $\mathbb{G}_{\mathbb{Q}}$ satisfies these conditions. □

Metrically universal abelian group

Corollary

Let G be an abelian group with invariant metric of density κ. Then there exists an extension of the metric to $H = G \oplus (\bigoplus_{\alpha<\kappa\times\aleph_0} \mathbb{Z})$ such that $\bigoplus_{\alpha<\kappa\times\aleph_0} \mathbb{Z}$ is dense in H.

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Sketch of the proof. Let $\{g_\alpha : \alpha < \kappa\} \leq G$ be a dense subgroup of cardinality κ. Apply the Proposition $\kappa \times \aleph_0$-many times so that for every $\alpha < \kappa$ and $n \in \mathbb{N}$ there is some $h_{\alpha,n}$ such that

$$d(g_\alpha, h_{\alpha,n}) < 1/n$$

□

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Corollary

It follows that to prove that $\mathbb{G}$ is metrically universal, it suffices to prove that $\mathbb{G}$ contains copy of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ with every possible invariant metric.

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Proof. Let G be an arbitrary abelian separable group with invariant metric. According to the previous Corollary we get that there is a supergroup $H = G \oplus (\bigoplus_{n\in\mathbb{N}} \mathbb{Z})$ such that $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ is dense in H.

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Proof. Let G be an arbitrary abelian separable group with invariant metric. According to the previous Corollary we get that there is a supergroup $H = G \oplus (\bigoplus_{n\in\mathbb{N}}\mathbb{Z})$ such that $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}$ is dense in H. Thus since $\mathbb{G}$ is metrically complete, if $\mathbb{G}$ contains this dense subgroup of H it contains the completion of H which contains G. $\square$

New group structure on the Urysohn space

Proposition

$\mathbb{G}$ is isometrically isomorphic neither to the Shkarin/Niemiec group nor to the Cameron/Vershik group.

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$\mathbb{G}$ is isometrically isomorphic neither to the Shkarin/Niemiec group nor to the Cameron/Vershik group.

Sketch of the proof. For every abelian group G with invariant metric and any element $g \in G$ consider the sequence:

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All the machinery from the countable case generalizes without substantial change.

Theorem

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then there exists an abelian group $\mathbb{G}_\kappa$ of density κ with invariant metric which is metrically universal for the corresponding class. Moreover, it is isometric to the generalized Urysohn space of density κ.

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Theorem

$\mathcal{G}_N$ is again a (generalized) Fraïssé class.

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Proposition

The Fraïssé limit of $\mathcal{G}_N$, which is algebraically a free group with countably many generators, is isometric to the rational Urysohn space. It follows the completion is isometric to the Urysohn space.

Questions - universal groups

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- The class of all Polish groups admitting compatible two-sided invariant metric - open

Questions - universal groups

Conjecture

The completion of the Fraïssé limit of $\mathcal{G}_N$ is a universal Polish group admitting two-sided invariant metric (topologically but maybe even metrically).

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Question (Shkarin)

Does there exist a metrically universal separable group?

Questions - Fraïssé classes of groups

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Question (Niemiec)

Does there exist a metric group of bounded exponent (other than 2 and 3) that is isometric to the Urysohn space?

Questions - metrically universal abelian group

Question

Is the metrically universal abelian group $\mathbb{G}$ (apart from being universal) also ultrahomogeneous? That is, is it true that any isometric isomorphism between two finitely generated subgroups extends to an isometric automorphism of the whole group?
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Question

Can you avoid using the Fraïssé theory? Respectively, do such universal groups exist for every density?

Questions - Cameron-Vershik group

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If the Cameron-Vershik group does not admit non-trivial continuous characters, is it extremely amenable?

This is related to an important open question whether there exists a monothetic group without continuous characters that is extremely amenable (Glasner).

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These universal groups must be useful in (harmonic) analysis. Connect all this stuff with the 'mainstream' mathematics".