Structural properties of orderings on multisets

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Definition (Multisets)

The family of all multisets over a set X is denoted by $\mathcal{M}(X)$, e.g.

$$\mathcal{M}(X) = \{A \in \mathbb{N}^X : |supp(A)| < \aleph_0\},$$

where $supp(A) = \{x \in X : A(x) \neq 0\}$.

Definition (Dershowitz-Manna Ordering)

Assume that (X, R) is a binary relation system.
For $A, B \in \mathcal{M}(X)$ we put

$$A \; R^X_{mult} \; B$$

$$\Updownarrow$$

$$(A \neq B) \wedge (\forall x \in X)(A(x) > B(x) \rightarrow (\exists y \in Y)(x \; R \; y \wedge A(y) < B(y))).$$

We put

$$\mathcal{M}(X, R) = (\mathcal{M}(X), \; R^X_{mult}).$$

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Notation

Suppose that (X, R) and (Y, S) are two binary relation systems.

- if $X \cap Y = \emptyset$ then $(X, R) \oplus (Y, S) = (X \cup Y, R \cup S)$
- if $X \cap Y = \emptyset$ then $(X, R) \triangleleft (Y, S) = (X \cup Y, R \cup S \cup (X \times Y))$

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- $(X, R) \otimes (Y, S) = (X \times Y, R \otimes S)$, where

$$(x, y) R \otimes S (x', y')$$

$$\Updownarrow$$

$$((x, y) \neq (x', y')) \wedge ((x = x') \vee (xRx')) \wedge ((y = y') \vee (ySy')).$$

- $(X, R) \otimes_{lex} (Y, S) = (X \times Y, R \otimes_{lex} S)$, where

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If (X, R) and (Y, S) are binary relation systems and $X \cap Y = \emptyset$ then

1. $\mathcal{M}((X, R) \oplus (Y, S)) \simeq \mathcal{M}(X, R) \otimes \mathcal{M}(Y, S)$
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Corollary

If α is an ordinal number then $\mathcal{M}(\alpha, \in) \simeq (\omega^\alpha, \in)$.

Proof

$\mathcal{M}(0, \in) = (\emptyset, \in)$ and $\mathcal{M}(1, \in) \simeq (\omega, \in)$.

$$\mathcal{M}(\alpha + 1, \in) \simeq \mathcal{M}((\alpha, \in) \triangleleft (1, \in)) \simeq \mathcal{M}(1, \in) \otimes_{lex} \mathcal{M}(\alpha, \in),$$

so

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Corollary [Dershowitz, Manna]

Suppose that (X, R) is a well-founded binary relation system. Then $\mathcal{M}(X, R)$ is a well-founded binary relation system.

Definition (Well quasi-ordering)

A quasi-ordering $(Q, \leq)$ is a *well-quasi-ordering* (wqo) if for every infinite sequence $a_1, a_2, a_3, \ldots$ from Q there exist $i < j \in \mathbb{N}$ such that $a_i \leq a_j$.

Remark

Assume that $(X, \leq)$ is a quasi-order. TFAAE:

1. $(X, \leq)$ is wqo.
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Theorem

Assume that partial ordering (X, R) is a well-quasi-ordering. Then $\mathcal{M}(X, R)$ is a well-quasi ordering, too.

Proof

Suppose $(\mathcal{M}(X),\ R^X_{mult}\ )$ is not a well-quasi-ordering.

There is a one-to-one sequence $f_n : X \to \mathbb{N}$ of elements of $\mathcal{M}(X)$ such that for $i < j$ we have that $\neg f_i\ R^X_{mult}\ f_j$.

Let us define

$$X_i^j = \{x \in X : f_i(x) > f_j(x) \wedge (\forall y)(xRy \to f_i(y) \geq f_j(y))\}.$$

$0 < |X_i^j| < \omega$.

Let x_i^j be any R-maximal element of X_i^j.

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For $i < j$

$$f_i(x_i^j) > f_j(x_i^j) \text{ and } \forall y \; x_i^j R y \rightarrow f_i(y) = f_j(y).$$

For $n < i, j$ and y such that $x_n^i = x_n^j R y$

$$f_i(y) = f_j(y) = f_n(y).$$

Consider the set $X_0 = \{x_0^j : j > 0\}$. Since it is a subset of $supp f_0$, it is a finite set.
Define a_0 to be an element of X_0 such that $A_0 = \{j : x_0^j = a_0\}$ is infinite.

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In the n-th step of construction we have a finite sequence $(a_0, a_1, \ldots, a_{n-1})$ and a sequence of infinite sets $\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{n-1}$ such that $\forall i < n \; A_{n-1} \subseteq \{j : a_i = x^j_{\min A_i}\}$.

Consider $X_n = \{x^j_{\min A_{n-1}} : j \in A_{n-1}\} \subseteq \mathit{suppf}_{\min A_{n-1}}$.

Define $a_n \in X_n$ and $A_n \subseteq A_{n-1}$ in the way that $A_n = \{j \in A_{n-1} : x^j_{\min A_{n-1}} = a_n\}$ is infinite.

Finally we get a sequence (a_n) which witnesses that (X, R) is not a well-quasi-ordering, since for $i < j$ we have that $\neg a_i R a_j$.

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Theorem

Suppose that $(X, <)$ is a dense linear ordering without minimal element. Then $\mathcal{M}(X, <)$ is a dense linear ordering, too.

Corollary

$\mathcal{M}(\mathbb{Q}, <) \simeq (\mathbb{Q}^{\geq 0}, <)$

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