Span and Chainability in Non-metric Continua

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DEFINITION  Continuum = compact Hausdorff connected topological space.

DEFINITION  Let $X$ be a continuum. A chain is a nonempty, finite collection $C = \{C_1, \ldots, C_n\}$ of open subsets $C_i$ of $X$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements $C_i$ of $C$ are called links of the chain $C$.

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**DEFINITION** A continuum $X$ is chainable if every open cover has an open cover refinement which is a chain.
DEFINITION A continuum $X$ has span zero if every subcontinuum $Z$ of $X \times X$, which projects onto the same set on both coordinates, has a nonempty intersection with the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ of $X$. Otherwise we say that $X$ has span non-zero.
Lelek’s conjecture

THEOREM (Lelek 1964) Every chainable continuum has span zero.

CONJECTURE (Lelek) Continuum having span zero is chainable.

OUR RESULT If there is a non-metric counterexample, there is also a metric counterexample.
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DEFINITION A lattice is called disjunctive if it models the following sentence

$$\forall ab \exists c \ (a \nleq b \rightarrow c \neq 0 \text{ and } c \leq a \text{ and } b \land c = 0).$$
Wallman’s representation theorem

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**DEFINITION** A lattice is called **normal** if it models the following sentence

$$\forall ab \exists cd \ (a \wedge b = 0 \rightarrow a \wedge d = 0 \text{ and } b \wedge c = 0 \text{ and } c \lor d = 1).$$
THEOREM (Wallman 1938) Let $L$ be a distributive disjunctive normal lattice. Then there is a compact Hausdorff space $wL$ with a base for closed sets being isomorphic to $L$. The points of $wL$ are the ultrafilters on $L$. The sets $U(a) = \{ x \in wL | a \in x \}$ form a base for closed sets for the topology on $wL$. Wallman’s representation extends to lattice homomorphisms and provides a functor $w$. 

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Wallman’s representation extends to lattice homomorphisms and provides a functor $w$. 
**Ultracopower**

DEFINITION \( q : X \times I \to I \) projection, \( I \) discrete

\[ \beta(q) : \beta(X \times I) \to \beta(I) - \text{Čech-Stone lifting of } q \]

Ultracopower \( \sum U X \) of \( X \) with respect to an ultrafilter \( U \) on \( I \) is \((\beta(q))^{-1}[U]\).
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**LEMMA**  
$B$ a lattice base for $X \rightarrow \sum_{U} X = w(\prod_{U} B)$.  

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codiagonal map \( \nabla \equiv \beta(p) |_{\sum \_U X} \)
Ultracopower $\sum_{\mathcal{U}} X$ of $X$ with respect to an ultrafilter $\mathcal{U}$ on $I$ is $(\beta(q))^{-1}[\mathcal{U}]$. 

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**LEMMA** Let $\Delta : B \to \prod_{\mathcal{U}} B$ be the diagonal embedding of a distributive disjunctive normal lattice $B$ to its ultrapower. Then $w(\Delta) = \nabla$. 

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Elementarity

Fix a first-order language $\mathcal{L}$.

**DEFINITION**

A and $B$ - $\mathcal{L}$-structures. $B$ is an *elementary substructure* of $A$ if $B$ is a substructure of $A$ and for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in B$, $B |_{a_1, \ldots, a_n} = \phi$ if and only if $A |_{a_1, \ldots, a_n} = \phi$.

**ŁOWENHEIM-SKOLEM THEOREM**

Let $A$ be an infinite $\mathcal{L}$-structure and let $X \subset A$. Denote $\kappa = \max(|\mathcal{L}|, |X|)$. Then for every cardinal $\lambda$ such that $\kappa \leq \lambda \leq |A|$, there exists an elementary substructure $B$ of $A$ such that $X \subset B$ and $|B| = \lambda$.
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\[ B \models \phi[a_1, \ldots, a_n] \text{ if and only if } A \models \phi[a_1, \ldots, a_n]. \]
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**LÖwenheim-Skolem Theorem** Let $A$ be an infinite $\mathcal{L}$-structure and let $X \subset A$. Denote $\kappa = \max(|\mathcal{L}|, |X|)$. Then for every cardinal $\lambda$ such that $\kappa \leq \lambda \leq |A|$, there exists an elementary substructure $B$ of $A$ such that $X \subset B$ and $|B| = \lambda$. 

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Elementarity in set theory

For a cardinal $\theta$, $H(\theta)$ denotes the set of all sets whose transitive closure has cardinality less than $\theta$. These sets are very important and useful because if $\theta$ is uncountable regular then $H(\theta) \models \text{ZFC - P}$. If $M$ is an elementary submodel of $H(\theta)$ such that $2^X \in M$ then $L = M \cap 2^X$ is an elementary sublattice of $2^X$. Similarly $K = M \cap 2^X \times X$ is an elementary sublattice of $2^X \times X$. 

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If $\mathcal{M}$ is an elementary submodel of $H(\theta)$ such that $2^X \in \mathcal{M}$ then $L = \mathcal{M} \cap 2^X$ is an elementary sublattice of $2^X$. Similarly, $K = \mathcal{M} \cap 2^{X \times X}$ is an elementary sublattice of $2^{X \times X}$. 
Applying elementarity

THEOREM (van der Steeg 2003) \( wK \cong wL \times wL \).
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THEOREM (van der Steeg 2003) \[ w_K \cong w_L \times w_L \]

THEOREM (van der Steeg 2003) \( X \) is chainable if and only if \( w_L \) is chainable.
Let $\kappa$ be a cardinal, $\lambda = \min\{\mu \mid \kappa^\mu > \kappa\}$ and let $A$ and $B$ be two elementarily equivalent $L$-structures with $\text{card}(A), \text{card}(B) < \lambda$. Then there exists an ultrafilter $U$ over $\kappa$ such that $\prod_U A$ and $\prod_U B$ are isomorphic.
THEOREM (DB+KPH 2008) If $X$ is a continuum having span zero, then $wL$ has span zero as well.
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Proof

$$
\begin{array}{ccc}
\Delta & & \\
\Pi_u K & \xrightarrow{e} & 2^X \times X \\
\Delta & & \\
\Pi_u 2^X \times X & \xrightarrow{h} & \Pi_u 2^X \times X
\end{array}
$$
\[ \sum_U wK \leftarrow^{w(h)} \sum_U X \times X \]

\[ wL \times wL \cong wK \leftarrow^{w(e)} X \times X \]

(2)
Proof

$$w_L \times w_L \cong w_K \xleftarrow{w(e)} X \times X$$

$$\sum_U w_K \leftarrow \sum_U X \times X$$

$$Z' = \nabla \circ w(h)^{-1} \left[ \sum_U Z \right].$$
Questions

**Question 1**  Is there an easier (more direct) proof of the reflection of span zero?
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Question 2 If \( L \) is an elementary sublattice of \( 2^X \), is the Wallman representation of the elementary embedding of \( L \) into \( 2^X \) confluent.
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Question 1 Is there an easier (more direct) proof of the reflection of span zero?

Question 2 If $L$ is an elementary sublattice of $2^X$, is the Wallman representation of the elementary embedding of $L$ into $2^X$ confluent.

Question 3 Is there a (non)-metric continuum that has span zero and is not chainable?
THANK YOU!!!