On additivity of permitted sets

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Winterschool on Abstract Analysis, Hejnice, 2009

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We will work on the unit circle ("torus") $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
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- P.E. (2004) proved that every permitted set is perfectly meager.

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3. Is $\{X \subseteq \mathbb{T} : X \text{ is permitted}\}$ a σ-ideal?

Combinatorics of Arbault sets

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Denote $TS = \left\{a \in \omega^\omega : \frac{a_n}{a_{n+1}} \to 0\right\}$.

Every Arbault set is included in a set of the form $A(a)$ for some $a \in TS$.

Characterization of permitted sets

Corollary

A set $X \subseteq \mathbb{T}$ *is permitted iff for every* $a \in TS$ *there exists a bounded matrix* $z \in \mathbb{Z}^{\omega \times \omega}$ *such that* $X \subseteq A(b)$ *where* $b = z \cdot a$*.*

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For $a, b \in TS$ and $k \in \omega$, let

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Problem

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