On Bernstein sets, κ-coverings and quotient groups

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Definition

- A set $B \subseteq \mathbb{R}$ is a Bernstein set, if for every perfect set $P \subseteq \mathbb{R}$

$$B \cap P \neq \emptyset \ \wedge \ P \nsubseteq B,$$

- A Bernstein group is a subgroup of $\mathbb{R}$ which is a Bernstein set.

Definition

- A set $C \subseteq \mathbb{R}$ is a κ-covering, if for every set $X \in [\mathbb{R}]^\kappa$ there exists $t \in \mathbb{R}$ such that $t + X \subseteq C$.
- A set C is a $< \kappa$-covering, if it is a λ-covering, for every $\lambda < \kappa$.

Theorem (Kraszewski–Ratowski–Szczepaniak–Żeberski)

There exists a partition of $\mathbb{R}$ *into* $\mathfrak{c}$ *many Bernstein sets, each of them being a* $< cf(\mathfrak{c})$*-covering.*

Question (Kraszewski–Ratowski–Szczepaniak–Żeberski)

Assume that $\mathfrak{c} > cf(\mathfrak{c}) = \omega_1$. *Does there exist a Bernstein set which is an* ω_1*-covering?*

Quotient groups and κ-coverings

Proposition

Assume that $G \lhd \mathbb{R}$ *is a subgroup and let*

$$\pi : \mathbb{R} \to \mathbb{R}/G$$

be the quotient epimorphism, i.e.

$$\pi(x) = [x]_G = x + G.$$

Then the following conditions are equivalent, for a set $C \subseteq \mathbb{R}/G$

- C *is a* κ*-covering in* $\mathbb{R}/G$*,*
- $\pi^{-1}[C]$ *is a* κ*-covering in* $\mathbb{R}$*.*

Quotient groups and Bernstein sets

Lemma

For every κ such that $\omega \leq \kappa \leq \mathfrak{c}$, there exists a subgroup $B \lhd \mathbb{R}$ such that

- *B is a Bernstein group,*
- $\left|\mathbb{R}/B\right| = \kappa$.

Proof.

- Construct disjoint Bernstein sets B_0, B_1 such that $B_0 \cup B_1$ is linearly independent over $\mathbb{Q}$,
- Find a Hamel base $H \supseteq B_0 \cup B_1$ and $Z \subseteq B_1$ of cardinality κ,
- Let $B = span(H \setminus Z)$, then $\mathbb{R}/B \simeq span(Z)$,
- B intersects all perfect sets, because $B_0 \subseteq B$ does,
- $P \not\subseteq B$, for a perfect set P – otherwise $z + P$, for $z \in Z$, would be disjoint with B. □

Remarks on the lemma

Remark

The group $\mathbb{R}/B$ is isomorphic to the $|Z|$-dimensional linear space over $\mathbb{Q}$. Thus:

- if $|Z| = 1$, then $\mathbb{R}/B \simeq \mathbb{Q}$,
- if $|Z| = \mathfrak{c}$, then $\mathbb{R}/B \simeq \mathbb{R}$.

Remark

Let $B \lhd \mathbb{R}$ be a Bernstein group and let $\emptyset \neq \mathcal{A} \subsetneq \mathbb{R}/B$. Then $\bigcup \mathcal{A}$ is a Bernstein set.

The answer

Theorem

For every κ such that $\omega \leq \kappa \leq \mathfrak{c}$, there exists a Bernstein set which is a $< \kappa$-covering and is not a κ-covering.

Corollary

There exists a Bernstein set which is a $< \mathfrak{c}$-covering. In particular, a Bernstein ω_1-covering exists, if and only if, $\mathfrak{c} > \omega_1$.

Corollary

There exists a $< \mathfrak{c}$-covering which is completely nonmeasurable with respect to every σ-algebra of the form $\mathcal{B}or[\mathcal{I}]$, where $\mathcal{I}$ is a σ-ideal with co-analytic base.

Bernsteins for free

Theorem

Suppose that $\{A_\xi : \xi < \kappa\}$ *is a partition of* $\mathbb{R}$, $\kappa > 1$, *and let*

$$\lambda_\xi = \min\{\lambda \in \mathit{Card} : A_\xi \textit{ is not a } \lambda\textit{-covering}\}.$$

Then there exists a partition $\{B_\xi : \xi < \kappa\}$ *of* $\mathbb{R}$ *into Bernstein sets such that*

$$\lambda_\xi = \min\{\lambda \in \mathit{Card} : B_\xi \textit{ is not a } \lambda\textit{-covering}\}.$$

Proof.

- find a Bernstein group $B \lhd \mathbb{R}$ with $\mathbb{R}/B \simeq \mathbb{R}$,
- consider a partition $\{A'_\xi : \xi < \kappa\}$ of $\mathbb{R}/B$ with the same characteristics as $\{A_\xi : \xi < \kappa\}$,
- put $B_\xi = \pi^{-1}[A'_\xi]$.

□

Reproving results from Wrocław

Theorem (Kraszewski–Ralowski–Szczepaniak–Żeberski)

There exists a partition of $\mathbb{R}$ into Bernstein sets A, B, none of them being a 2-covering.

Proof.

$R_0 = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1)$, $R_1 = \bigcup_{k\in\mathbb{Z}}[2k-1, 2k)$ is a partition of $\mathbb{R}$ into sets which are not 2-coverings. □

Coverings and partitions

Theorem

There exists a Bernstein set which is a $<\mathfrak{c}$-covering.

Theorem (Kraszewski–Rałowski–Szczepaniak–Żeberski)

There exists a partition of $\mathbb{R}$ into $\mathfrak{c}$ many Bernstein sets, each of them being a $<cf(\mathfrak{c})$-covering.

Corollary (Kraszewski–Rałowski–Szczepaniak–Żeberski)

If $\mathfrak{c}$ is regular, then there exists a partition of $\mathbb{R}$ into $\mathfrak{c}$ many $<\mathfrak{c}$-coverings which are Bernstein sets.

Lemma (Kraszewski–Rałowski–Szczepaniak–Żeberski)

If G is an abelian group of regular cardinality λ, then there exists a partition of G into λ many $< \lambda$-coverings.

Theorem

For every $\kappa < \mathfrak{c}$, there exists a partition of $\mathbb{R}$ into κ^+ many κ-coverings which are Bernstein sets.

Proof.

- take a Bernstein group B with $|\mathbb{R}/B| = \kappa^+$,
- find a partition of $\mathbb{R}/B$ into κ^+ many κ-coverings,
- get Bernsteins for free.

□

Singular continuum - a negative result

Theorem

If there exist two disjoint $<\mathfrak{c}$-coverings in $\mathbb{R}$, then $\mathfrak{c}$ is regular.

Lemma

If $B \subseteq \mathbb{R}$ is a $<\mathfrak{c}$-covering, then fewer than $\mathfrak{c}$ translates of its complement do not cover $\mathbb{R}$.

Proof of the theorem.

- let A, B be disjoint $<\mathfrak{c}$-coverings, $A \cup B = \mathbb{R}$,
- fewer than $\mathfrak{c}$ translates of A do not cover $\mathbb{R}$,
- let $\mathbb{R} = \bigcup_{\xi < cf(\mathfrak{c})} X_\xi$, with $|X_\xi| < \mathfrak{c}$,
- find t_ξ such that $t_\xi + A \supseteq X_\xi$,
- $\bigcup_{\xi < cf(\mathfrak{c})} (t_\xi + A) = \mathbb{R}$, so $cf(\mathfrak{c}) = \mathfrak{c}$. □