Complete pairs of coanalytic sets

H. Michalewski

Institute of Mathematics
University of Warsaw

Winter School, Hejnice 2009
Summary

1. **Borel, analytic and coanalytic sets**

2. **Definition of a complete pair**

3. **Basic examples of complete pairs**

4. **A complete pair in the space of continuous functions**

5. **A complete pair in the theory of automata**
Summary

1. Borel, analytic and coanalytic sets

2. Definition of a complete pair

3. Basic examples of complete pairs

4. A complete pair in the space of continuous functions

5. A complete pair in the theory of automata
Summary

1. Borel, analytic and coanalytic sets
2. Definition of a complete pair
3. Basic examples of complete pairs
4. A complete pair in the space of continuous functions
5. A complete pair in the theory of automata
Summary

1. Borel, analytic and coanalytic sets
2. Definition of a complete pair
3. Basic examples of complete pairs
4. A complete pair in the space of continuous functions
5. A complete pair in the theory of automata
Summary

1. Borel, analytic and coanalytic sets
2. Definition of a complete pair
3. Basic examples of complete pairs
4. A complete pair in the space of continuous functions
5. A complete pair in the theory of automata
**Definition**

*X is a **Polish space** if X is separable and completely metrizable.*

Cantor set $\mathcal{C}$, the reals $\mathbb{R}$, the naturals $\mathbb{N}$, the Banach space $C([0, 1])$ with $\|\cdot\|_\infty$ are all examples of Polish spaces.
**Definition**

A continuous function $f$ from a Polish space $X$ to a Polish space $Y$ is called a **complete pair** if $f(X)$ is a complete subset of $Y$.

The Cantor set $C$, the reals $\mathbb{R}$, the naturals $\mathbb{N}$, the Banach space $C([0, 1])$ with $\| \cdot \|_\infty$ are all examples of Polish spaces.
Definition

The Borel sets $\mathcal{B}(X)$ in a given topological space is the smallest $\sigma$–field containing all open sets of $X$. 
Definition

A set $A \subseteq X$ in a Polish space $X$ is **analytic** if there exists a Polish space $Y$ and a Borel set $B \subseteq X \times Y$ such that

$$A = \{ x \in X : \exists y \in Y \ (x, y) \in B \}.$$ 

Definition

A set $A \subseteq X$ in a Polish space $X$ is **coanalytic** if $X \setminus A$ is an analytic set.
A set $A \subseteq X$ in a Polish space $X$ is **analytic** if there exists a Polish space $Y$ and a Borel set $B \subseteq X \times Y$ such that

$$A = \{ x \in X : \exists y \in Y \langle x, y \rangle \in B \}.$$ 

A set $A \subseteq X$ in a Polish space $X$ is **coanalytic** if $X \setminus A$ is an analytic set.
**Definition**

A set $A$ in a topological space $X$ is **Wadge reducible** to a set $B$ in a topological space $Y$ if there exists a continuous mapping $\phi : X \to Y$ such that $A = \phi^{-1}[B]$.

**Definition**

A disjoint pair $A, B$ in a topological space $X$ is Wadge reducible to a disjoint $C, D$ in a topological space $Y$, if there exists a continuous mapping $\phi : X \to Y$ such that $A \leq_{\phi} C$ and $B \leq_{\phi} D$, that is $A = \phi^{-1}[C]$ and $B = \phi^{-1}[D]$. 
Definition

A set $A$ in a topological space $X$ is **Wadge reducible** to a set $B$ in a topological space $Y$ if there exists a continuous mapping $\phi : X \rightarrow Y$ such that $A = \phi^{-1}[B]$.

Definition

A disjoint pair $A, B$ in a topological space $X$ is Wadge reducible to a disjoint $C, D$ in a topological space $Y$, if there exists a continuous mapping $\phi : X \rightarrow Y$ such that $A \leq_{\phi} C$ and $B \leq_{\phi} D$, that is $A = \phi^{-1}[C]$ and $B = \phi^{-1}[D]$.
Definition

A set $A$ in a topological space $X$ is **Wadge reducible** to a set $B$ in a topological space $Y$ if there exists a continuous mapping $\phi : X \rightarrow Y$ such that $A = \phi^{-1}[B]$.

Definition

A disjoint pair $A, B$ in a topological space $X$ is **Wadge reducible** to a disjoint $C, D$ in a topological space $Y$, if there exists a continuous mapping $\phi : X \rightarrow Y$ such that $A \leq_{\phi} C$ and $B \leq_{\phi} D$, that is $A = \phi^{-1}[C]$ and $B = \phi^{-1}[D]$.
Definition

A set $A$ in a topological space $X$ is **Wadge reducible** to a set $B$ in a topological space $Y$ if there exists a continuous mapping $\phi : X \to Y$ such that $A = \phi^{-1}[B]$.

Definition

A disjoint pair $A, B$ in a topological space $X$ is **Wadge reducible** to a disjoint $C, D$ in a topological space $Y$, if there exists a continuous mapping $\phi : X \to Y$ such that $A \leq_\phi C$ and $B \leq_\phi D$, that is $A = \phi^{-1}[C]$ and $B = \phi^{-1}[D]$. 
Definition

A set $A$ in a topological space $X$ is **Wadge reducible** to a set $B$ in a topological space $Y$ if there exists a continuous mapping $\phi : X \to Y$ such that $A = \phi^{-1}[B]$.

Definition

A **disjoint pair** $A, B$ in a topological space $X$ is **Wadge reducible** to a disjoint $C, D$ in a topological space $Y$, if there exists a continuous mapping $\phi : X \to Y$ such that $A \leq_{\phi} C$ and $B \leq_{\phi} D$, that is $A = \phi^{-1}[C]$ and $B = \phi^{-1}[D]$. 
Definition

A disjoint pair of coanalytic sets $C, D$ in a Polish space $X$ is **complete**, if for every disjoint pair of coanalytic sets $A, B$ in the Cantor set the pair $A, B$ is Wadge reducible to the pair $C, D$.

The pair $C, D$ represents all essential properties of pairs of coanalytic sets. For example, in the class of coanalytic sets there exists a pair $A, B$ not separable by a Borel set. The same holds for all complete pairs.
Definition

A disjoint pair of coanalytic sets $C, D$ in a Polish space $X$ is **complete**, if for every disjoint pair of coanalytic sets $A, B$ in the Cantor set the pair $A, B$ is Wadge reducible to the pair $C, D$.

The pair $C, D$ represents all essential properties of pairs of coanalytic sets. For example, in the class of coanalytic sets there exists a pair $A, B$ not separable by a Borel set. The same holds for all complete pairs.
**Definition**

A disjoint pair of coanalytic sets $C, D$ in a Polish space $X$ is **complete**, if for every disjoint pair of coanalytic sets $A, B$ in the Cantor set the pair $A, B$ is Wadge reducible to the pair $C, D$.

The pair $C, D$ represents all essential properties of pairs of coanalytic sets. For example, in the class of coanalytic sets there exists a pair $A, B$ not separable by a Borel set. The same holds for all complete pairs.
Definition

A disjoint pair of coanalytic sets $C, D$ in a Polish space $X$ is **complete**, if for every disjoint pair of coanalytic sets $A, B$ in the Cantor set the pair $A, B$ is Wadge reducible to the pair $C, D$.

The pair $C, D$ represents all essential properties of pairs of coanalytic sets. For example, in the class of coanalytic sets there exists a pair $A, B$ not separable by a Borel set. The same holds for all complete pairs.
**Definition**

A disjoint pair of coanalytic sets $C, D$ in a Polish space $X$ is **complete**, if for every disjoint pair of coanalytic sets $A, B$ in the Cantor set the pair $A, B$ is Wadge reducible to the pair $C, D$.

The pair $C, D$ represents all essential properties of pairs of coanalytic sets. For example, in the class of coanalytic sets there exists a pair $A, B$ not separable by a Borel set. The same holds for all complete pairs.
In order to prove that a given disjoint pair $C, D$ of coanalytic sets is complete, it is enough to find a complete pair $A, B$ and a reduction $\phi$ such that $A \leq_{\phi} C$ and $B \leq_{\phi} D$. 
**Definition**

\[ T \subseteq \omega^{<\omega} \text{ is a tree, if } T \text{ is closed with respect to initial segments, that is for every } s \in T \text{ and an initial segment } r \preceq s \text{ we have } r \in T. \]

A sequence \( x \in \omega^\omega \) is a **branch** of \( T \), if for every \( n \in \omega \) we have \( x|n \in T \).
Definition

$T \subseteq \omega^\omega$ is a **tree**, if $T$ is closed with respect to initial segments, that is for every $s \in T$ and an initial segment $r \preceq s$ we have $r \in T$. A sequence $x \in \omega^\omega$ is a **branch** of $T$, if for every $n \in \omega$ we have $x|n \in T$. 
Definition

Let $\text{Tr} \subseteq 2^{\omega < \omega}$ be the set of all trees. We define $\text{WF}$ as the set of all well-founded trees and $\text{UB}$ as the set of all trees with exactly one branch.

J. Saint Raymond proved in 2007 that the pair $\text{WF}$, $\text{UB}$ is a complete pair of coanalytic sets.
Definition

Let $\text{Tr} \subseteq 2^{\omega^*}$ be the set of all trees. We define $\text{WF}$ as the set of all \textbf{well–founded trees} and $\text{UB}$ as the set of all trees with exactly one branch.

J. Saint Raymond proved in 2007 that the pair $\text{WF}, \text{UB}$ is a complete pair of coanalytic sets.
Definition

Let $\text{Tr} \subset 2^{\omega^*} \omega$ be the set of all trees. We define WF as the set of all well-founded trees and UB as the set of all trees with exactly one branch.

J. Saint Raymond proved in 2007 that the pair WF, UB is a complete pair of coanalytic sets.
Definition

Let $\text{Tr} \subset 2^{\omega<\omega}$ be the set of all trees. We define $\text{WF}$ as the set of all well-founded trees and $\text{UB}$ as the set of all trees with exactly one branch.

J. Saint Raymond proved in 2007 that the pair $\text{WF}$, $\text{UB}$ is a complete pair of coanalytic sets.
Definition

Every tree $T \in \text{WF}$ admits a natural rank $\text{rk}(T)$, which is an ordinal below $\omega_1$. Firstly we define inductively rank of $T$ for every vertex of $T$ and then define rank of $T$ as the rank of $\emptyset \in T$. If $T$ is not in WF, we define $\text{rk}(T)$ as $\omega_1$. 
Definition

Every tree $T \in \text{WF}$ admits a natural rank $\text{rk}(T)$, which is an ordinal below $\omega_1$. Firstly we define inductively rank of $T$ for every vertex of $T$ and then define rank of $T$ as the rank of $\emptyset \in T$. If $T$ is not in $\text{WF}$, we define $\text{rk}(T)$ as $\omega_1$. 
Definition

Every tree \( T \in WF \) admits a natural rank \( \text{rk}(T) \), which is an ordinal below \( \omega_1 \). Firstly we define inductively rank of \( T \) for every vertex of \( T \) and then define rank of \( T \) as the rank of \( \emptyset \in T \). If \( T \) is not in \( WF \), we define \( \text{rk}(T) \) as \( \omega_1 \).
Definition

Every tree \( T \in \text{WF} \) admits a natural rank \( \text{rk}(T) \), which is an ordinal below \( \omega_1 \). Firstly we define inductively rank of \( T \) for every vertex of \( T \) and then define rank of \( T \) as the rank of \( \emptyset \in T \). If \( T \) is not in \( \text{WF} \), we define \( \text{rk}(T) \) as \( \omega_1 \).
Definition

Let

\[ V_0 = \{ \langle S, T \rangle : S \in \text{WF}, \ rk(S) < rk(T) \} \]

and

\[ V_1 = \{ \langle S, T \rangle : T \in \text{WF}, \ rk(T) \leq rk(S) \}. \]

The sets \( V_0 \) i \( V_1 \) are disjoint and coanalytic and forms a complete pair.
Definition

Let

\[ V_0 = \{ \langle S, T \rangle : S \in \text{WF}, \ rk(S) < rk(T) \} \]

and

\[ V_1 = \{ \langle S, T \rangle : T \in \text{WF}, \ rk(T) \leq rk(S) \} \].

The sets \( V_0 \) i \( V_1 \) are disjoint and coanalytic and forms a complete pair.
Definition

Let

\[ V_0 = \{ \langle S, T \rangle : S \in \text{WF}, \, \text{rk}(S) < \text{rk}(T) \} \]

and

\[ V_1 = \{ \langle S, T \rangle : T \in \text{WF}, \, \text{rk}(T) \leq \text{rk}(S) \} . \]

The sets \( V_0 \) and \( V_1 \) are disjoint and coanalytic and forms a complete pair.
Definition

We define $\text{Diff}$ as a subset of $\mathcal{C}([0,1])$ consisting of all differentiable functions on the unit interval $[0,1]$.

In 1936 S. Mazurkiewicz proved that the set $\text{Diff}$ is a coanalytic non–Borel subset $\mathcal{C}([0,1])$.

Definition

Let $\text{Diff}_1$ be the set of all functions in $\mathcal{C}([0,1])$ which are not differentiable in exactly one point of $[0,1]$.

The Mazurkiewicz’s proof gives completeness of the pair $\text{Diff}, \text{Diff}_1$. 
Definition

We define **Diff** as a subset of $C([0, 1])$ consisting of all *differentiable functions* on the unit interval $[0, 1]$.

In 1936 S. Mazurkiewicz proved that the set Diff is an coanalytic non–Borel subset $C([0, 1])$.

Definition

Let **Diff$_1$** be the set of all functions in $C([0, 1])$ which are *not* differentiable in exactly one point of $[0, 1]$.

The Mazurkiewicz’s proof gives completeness of the pair Diff, Diff$_1$. 
Definition

We define $\text{Diff}$ as a subset of $C([0, 1])$ consisting of all differentiable functions on the unit interval $[0, 1]$.

In 1936 S. Mazurkiewicz proved that the set $\text{Diff}$ is an coanalytic non–Borel subset $C([0, 1])$.

Definition

Let $\text{Diff}_1$ be the set of all functions in $C([0, 1])$ which are not differentiable in exactly one point of $[0, 1]$.

The Mazurkiewicz’s proof gives completeness of the pair $\text{Diff}, \text{Diff}_1$. 
Definition

We define $\text{Diff}$ as a subset of $C([0,1])$ consisting of all differentiable functions on the unit interval $[0,1]$.

In 1936 S. Mazurkiewicz proved that the set $\text{Diff}$ is a coanalytic non–Borel subset $C([0,1])$.

Definition

Let $\text{Diff}_1$ be the set of all functions in $C([0,1])$ which are not differentiable in exactly one point of $[0,1]$.

The Mazurkiewicz’s proof gives completeness of the pair $\text{Diff}, \text{Diff}_1$. 
Let $S$ be the set of all **full binary trees** with vertices **labeled** by elements of the set $\{\exists, \forall\} \times \{0, 1\}$. Let $t \in S$.

From a vertex of $t$ one may go either right or left and the players $\exists$ and $\forall$ play a **game**, such that each of the players decides about a move from ‘his‘ vertices, that is from vertices labeled by $\exists$ and $\forall$ respectively.
Let $S$ be the set of all **full binary trees** with vertices **labeled** by elements of the set $\{\exists, \forall\} \times \{0, 1\}$. Let $t \in S$.

From a vertex of $t$ one may go either right or left and the players $\exists$ and $\forall$ play a **game**, such that each of the players decides about a move from ‘his‘ vertices, that is from vertices labeled by $\exists$ and $\forall$ respectively.
Let \( S \) be the set of all \textbf{full binary trees} with vertices \textbf{labeled} by elements of the set \( \{\exists, \forall\} \times \{0, 1\} \). Let \( t \in S \).

From a vertex of \( t \) one may go either right or left and the players \( \exists \) and \( \forall \) play a \textbf{game}, such that each of the players decides about a move from ‘his‘ vertices, that is from vertices labeled by \( \exists \) and \( \forall \) respectively.
The player $\exists$ wins, if all vertices occurring in a given play, with except of finitely many, have label 0. The player $\forall$ wins, if all vertices occurring in a given play, with except of finitely many, has label 1.
The player $\exists$ wins, if all vertices occurring in a given play, with except of finitely many, have label 0. The player $\forall$ wins, if all vertices occurring in a given play, with except of finitely many, has label 1.
**Definition**

Let $W_{0,1}$ be the set of all trees in $S$, such that the player $\exists$ has a winning strategy and $W'_{0,1}$ be the set of all trees in $S$, such that the player $\forall$ has a winning strategy.

The pair $W_{0,1}, W'_{0,1}$ is a complete pair of coanalytic sets. Sz. Hummel proved in his Master Dissertation that the sets $W_{0,1}, W'_{0,1}$ are coanalytic and that the sets $W_{0,1}, W'_{0,1}$ are not separable by a Borel set. This results were incorporated into a joint paper by D. Niwiński, Sz. Hummel and H. Michalewski accepted for STACS 2009.
Definition

Let $W_{0,1}$ be the set of all trees in $S$, such that the player $\exists$ has a winning strategy and $W'_{0,1}$ be the set of all trees in $S$, such that the player $\forall$ has a winning strategy.

The pair $W_{0,1}$, $W'_{0,1}$ is a complete pair of coanalytic sets. Sz. Hummel proved in his Master Dissertation that the sets $W_{0,1}$, $W'_{0,1}$ are coanalytic and that the sets $W_{0,1}$, $W'_{0,1}$ are not separable by a Borel set. This results were incorporated into a joint paper by D. Niwiński, Sz. Hummel and H. Michalewski accepted for STACS 2009.
Definition

Let $W_{0,1}$ be the set of all trees in $S$, such that the player $\exists$ has a winning strategy and $W'_{0,1}$ be the set of all trees in $S$, such that the player $\forall$ has a winning strategy.

The pair $W_{0,1}, W'_{0,1}$ is a complete pair of coanalytic sets. Sz. Hummel proved in his Master Dissertation that the sets $W_{0,1}, W'_{0,1}$ are coanalytic and that the sets $W_{0,1}, W'_{0,1}$ are not separable by a Borel set. This results were incorporated into a joint paper by D. Niwiński, Sz. Hummel and H. Michalewski accepted for STACS 2009.
Definition

Let $W_{0,1}$ be the set of all trees in $S$, such that the player $\exists$ has a winning strategy and $W'_{0,1}$ be the set of all trees in $S$, such that the player $\forall$ has a winning strategy.

The pair $W_{0,1}$, $W'_{0,1}$ is a complete pair of coanalytic sets. Sz. Hummel proved in his Master Dissertation that the sets $W_{0,1}$, $W'_{0,1}$ are coanalytic and that the sets $W_{0,1}$, $W'_{0,1}$ are not separable by a Borel set.

This results were incorporated into a joint paper by D. Niwiński, Sz. Hummel and H. Michalewski accepted for STACS 2009.
Definition

*Let $W_{0,1}$ be the set of all trees in $S$, such that the player $\exists$ has a winning strategy* and $W'_{0,1}$ be the set of all trees in $S$, such that the player $\forall$ has a winning strategy.*

The pair $W_{0,1}$, $W'_{0,1}$ is a complete pair of coanalytic sets. Sz. Hummel proved in his Master Dissertation that the sets $W_{0,1}$, $W'_{0,1}$ are coanalytic and that the sets $W_{0,1}$, $W'_{0,1}$ are not separable by a Borel set. This results were incorporated into a joint paper by D. Niwiński, Sz. Hummel and H. Michalewski accepted for STACS 2009.
The set $W_{0,1}$ is an example of set accepted by a non–deterministic tree automaton. The automaton has the states 0, 1 and $T$, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

$$i \langle \forall j \rangle \xrightarrow{a} j, j,$$

$$i \langle \exists j \rangle \xrightarrow{a} j, T, \quad i \langle \exists j \rangle \xrightarrow{a} T, j$$

and

$$T \xrightarrow{a} T, T,$$

where $i, j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$. 
The set $W_{0,1}$ is an example of set accepted by a non-deterministic tree automaton. The automaton has the states 0, 1 and T, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

$$i \langle \forall j \rangle \rightarrow j, j,$$

$$i \langle \exists j \rangle \rightarrow j, T, \quad i \langle \exists j \rangle \rightarrow T, j$$

and

$$T \rightarrow T, T,$$

where $i, j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$. 
The set $W_{0,1}$ is an example of set accepted by a non–deterministic tree automaton. The automaton has the states 0, 1 and T, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

$$i \langle \forall j \rangle \rightarrow j, j,$$

$$i \langle \exists j \rangle \rightarrow j, T,$$

$$i \langle \exists j \rangle \rightarrow T, j$$

and

$$T \xrightarrow{a} T, T,$$

where $i, j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$. 
The set $W_{0,1}$ is an example of set accepted by a **non–deterministic tree automaton**. The automaton has the **states** $0, 1$ and $T$, works over the **alphabet** $\{\exists, \forall\} \times \{0, 1\}$ and has the following **transitions**:

\[ i \langle \forall j \rangle \rightarrow j, j, \]
\[ i \langle \exists j \rangle \rightarrow j, T, \]
\[ i \langle \exists j \rangle \rightarrow T, j \]
\[ T \overset{a}{\rightarrow} T, T, \]

where $i,j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$.
The set $W_{0,1}$ is an example of set accepted by a non-deterministic tree automaton. The automaton has the states 0, 1 and T, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

\[
i \xrightarrow{\langle \forall, j \rangle} j, j,
\]

\[
i \xrightarrow{\langle \exists, j \rangle} j, T, \quad i \xrightarrow{\langle \exists, j \rangle} T, j
\]

and

\[
T \xrightarrow{a} T, T,
\]

where $i, j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$. 

Michalewski  Complete pairs of coanalytic sets
The set $W_{0,1}$ is an example of set accepted by a non–deterministic tree automaton. The automaton has the states 0, 1 and T, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

\[
i \langle \forall j \rangle \rightarrow j, j,
\]

\[
i \langle \exists j \rangle \rightarrow j, T, \quad i \langle \exists j \rangle \rightarrow T, j
\]

and

\[
T \xrightarrow{a} T, T,
\]

where $i, j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$.
The set $W_{0,1}$ is an example of set accepted by a **non–deterministic tree automaton**. The automaton has the states 0, 1 and T, works over the alphabet $\{\exists, \forall\} \times \{0, 1\}$ and has the following transitions:

$$i \xrightarrow{\langle\forall,j\rangle} j,j,$$

$$i \xrightarrow{\langle\exists,j\rangle} j,T, \quad i \xrightarrow{\langle\exists,j\rangle} T,j$$

and

$$T \xrightarrow{a} T,T,$$

where $i,j \in \{0, 1\}$ and $a \in \{\exists, \forall\} \times \{0, 1\}$. 
The **rank** of the states 0 and T is 0 and the rank of the state 1 is 1. A tree \( t \in S \) is **recognized** by the automaton if there exists a run of the automaton such that on every branch \( x \) of \( t \) the \( \limsup \rho(x(n)) \) is even (in our case the only possible even rank is 0). The set \( W_{0,1}' \) is accepted by a very similar automaton, such that the roles of \( \exists \) and \( \forall \) are swapped and at the same time the roles of 0 and 1 are swapped.
The **rank** of the states 0 and T is 0 and the rank of the state 1 is 1. A tree $t \in S$ is **recognized** by the automaton if there exists a run of the automaton such that on every branch $x$ of $t$ the $\limsup \rho(x(n))$ is even (in our case the only possible even rank is 0). The set $W'_{0,1}$ is accepted by a very similar automaton, such that the roles of $\exists$ and $\forall$ are swapped and at the same time the roles of 0 and 1 are swapped.
The **rank** of the states 0 and T is 0 and the rank of the state 1 is 1. A tree $t \in S$ is **recognized** by the automaton if there exists a run of the automaton such that on every branch $x$ of $t$ the lim sup $\rho(x(n))$ is even (in our case the only possible even rank is 0). The set $W'_{0,1}$ is accepted by a very similar automaton, such that the roles of $\exists$ and $\forall$ are swapped and at the same time the roles of 0 and 1 are swapped.
An analogous definition gives sets $W_{i,k}, W'_{i,k}$ for larger sets of indices \{i, \ldots, n\}. One can prove, that the complement of the set $W_{0,1}$ is not recognized by an automaton with index \{0, 1\} but is accepted by an automaton with index \{1, 2\}.
An analogous definition gives sets $W_{i,k}, W'_{i,k}$ for larger sets of indices $\{i, \ldots, n\}$. One can prove, that the complement of the set $W_{0,1}$ is not recognized by an automaton with index $\{0, 1\}$ but is accepted by an automaton with index $\{1, 2\}$.
An analogous definition gives sets $W_{i,k}$, $W'_{i,k}$ for larger sets of indices $\{i, \ldots, n\}$. One can prove, that the complement of the set $W_{0,1}$ is not recognized by an automaton with index $\{0, 1\}$ but is accepted by an automaton with index $\{1, 2\}$. 
It was proved by M. O. Rabin, that every two disjoint sets of trees $A, B$ accepted by automata with indices $\{1, 2\}$ is possible to separate by set $C$, which is simultaneously accepted by automata with indices $\{1, 2\}$ and $\{0, 1\}$.

For larger set of indices the question of separation remains open.
It was proved by M. O. Rabin, that every two disjoint sets of trees $A$, $B$ accepted by automata with indices $\{1, 2\}$ is possible to separate by set $C$, which is simultaneously accepted by automata with indices $\{1, 2\}$ and $\{0, 1\}$.

For larger set of indices the question of separation remains open.
It was proved by M. O. Rabin, that every two disjoint sets of trees $A$, $B$ accepted by automata with indices $\{1, 2\}$ is possible to separate by set $C$, which is simultaneously accepted by automata with indices $\{1, 2\}$ and $\{0, 1\}$.

For larger set of indices the question of separation remains open.
Bibliography I

Sz. Hummel

Własności oddzielania zbiorów drzew definiowalnych przez automaty
Master Thesis prepared under supervision of D. Niwiński,
Institute of Informatics, University of Warsaw, 2008.

A. S. Kechris

Classical Descriptive Set Theory.

S. Mazurkiewicz

Über die Menge der differenzierbaren Funktionen.
Fund. Math. 27 (1936), 244–249.
M. O. Rabin.  
*Weakly definable relations and special automata.*  

W. Thomas.  
*Languages, automata, and logic.*  

J. Saint Raymond  
*Complete pairs of coanalytic sets.*  