GAMES ON BOOLEAN ALGEBRAS OF UNCOUNTABLE LENGTH

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Cut-and-choose games on Boolean algebras

Played by two players, White and Black, on a complete Boolean algebra $\mathbb{B}$. Games of type (κ, λ, μ) are played in κ-many moves:

First White chooses $p \in \mathbb{B}^+$.

In α-th move:

White cuts p into λ pieces (i.e. chooses a maximal antichain A_α below p of cardinality at most λ)

Black chooses $< \mu$ of those pieces (i.e. a subset $B_\alpha \subseteq A_\alpha$ of cardinality $< \mu$).

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A special case: $\lambda = \mu = 2$

In the α-th move White chooses $p_\alpha \in (0, p)_{\mathbb{B}}$ and Black chooses $i_\alpha \in \{0, 1\}$.

Thus they obtain a sequence $\langle p_0^{i_0}, \ldots, p_\alpha^{i_\alpha}, \ldots \rangle$, where

$$q^i = \begin{cases} q & \text{if } i = 0 \\ p \setminus q & \text{if } i = 1 \end{cases}$$

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The game $\mathcal{G}_{\text{dist}}$

Jech in [1] defined a game of type $(\omega, 2, 2)$: White wins the game $\langle p, p_0, i_0, \ldots, p_n, i_n, \ldots \rangle$ iff

$$\bigwedge_{n<\omega} p_n^{i_n} = 0.$$

Theorem

The following conditions are equivalent:
(a) $\mathbb{B}$ is not $(\omega, 2)$-distributive;
(b) In some generic extension $V_{\mathbb{B}}[G]$ there is a new function $f : \omega \longrightarrow 2$;
(c) White has a winning strategy in $\mathcal{G}_{\text{dist}}$ played on $\mathbb{B}$.

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A generalization

Dobrinen in [2] generalized it to a game $\mathcal{G}_{\rm dist}(\kappa,\lambda,\mu)$ of type (κ,λ,μ): White wins iff $\bigwedge_{\alpha<\kappa}\bigvee B_\alpha = 0$.

Theorem

(a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d), where
(a) $\mathbb{B}$ is not (κ,λ,μ)-distributive;
(b) there is $f:\kappa\to\lambda$ in some generic extension $V_{\mathbb{B}}[G]$ such that no $g:\kappa\to[\lambda]^{<\mu}$ in V is such that $f(\alpha)\in g(\alpha)$ for all $\alpha<\kappa$;
(c) White has a winning strategy in $\mathcal{G}_{\rm dist}(\kappa,\lambda,\mu)$ played on $\mathbb{B}$;
(d) $\mathbb{B}$ is not $((\lambda^{<\mu})^{<\kappa},\lambda,\mu)$-distributive.

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White has a winning strategy in $\mathcal{G}_{\rm dist}(\kappa,2,2)$ played on $\mathbb{B}$ iff $\mathbb{B}$ is not $(\kappa,2)$-distributive.

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The game $\mathcal{G}_{\mathrm{ls}}(\kappa)$

$\mathcal{G}_{\mathrm{ls}}(\kappa)$ is the game of type $(\kappa, 2, 2)$ in which White wins the game $\langle p, p_0, i_0, \ldots, p_\alpha, i_\alpha, \ldots \rangle$ iff

$$\bigwedge_{\beta<\kappa} \bigvee_{\alpha\geq\beta} p_\alpha^{i_\alpha} = 0.$$

Existence of a winning strategy for Black

Theorem

If $\mathbb{B}$ is a complete Boolean algebra and $\kappa \geq \pi(\mathbb{B})$, then Black has a winning strategy in the game $\mathcal{G}_{\mathrm{ls}}(\kappa)$ played on $\mathbb{B}$, where

$$\pi(\mathbb{B}) = \min\{\lambda : \mathbb{B} \text{ has a dense subset of cardinality } \lambda\}.$$

Theorem

If a complete Boolean algebra $\mathbb{B}$ contains a λ-closed dense subset $D \subseteq \mathbb{B}^+$, then for each infinite cardinal $\kappa < \lambda$ Black has a winning strategy in the game $\mathcal{G}_{\mathrm{ls}}(\kappa)$.

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(a)⇒(b)⇒(c), where
(a) In some generic extension, $V_{\mathbb{B}}[G]$, κ is a regular cardinal and the cardinal $(2^\kappa)^V$ is collapsed to κ;
(b) White has a winning strategy in the game $\mathcal{G}_{\mathrm{ls}}(\kappa)$ played on $\mathbb{B}$;
(c) in some generic extension, $V_{\mathbb{B}}[G]$, the sets $({}^\kappa 2)^V$ and $({}^{<\kappa}2)^V$ are of the same size.

(c)⇒(b) need not be true; an example: $\mathbb{B} =\mathrm{Col}(\aleph_1, \aleph_{\omega+1})$ in a model of MA$+2^{\aleph_0} = \aleph_{\omega+1}$, for $\kappa = \aleph_\omega$.

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Existence of a winning strategy for White (continued)

Corollary

White has a winning strategy in $\mathcal{G}_{\mathrm{ls}}(\omega)$ played on $\mathbb{B}$ iff forcing by $\mathbb{B}$ collapses $\mathfrak{c}$ to ω in some generic extension.

If White has a winning strategy in $\mathcal{G}_{\mathrm{ls}}(\kappa)$, then $\kappa \in [\mathfrak{h}_2(\mathbb{B}), \pi(\mathbb{B}))$, where

$$\mathfrak{h}_2(\mathbb{B}) = \min\{\lambda : \mathbb{B} \text{ is not } (\lambda, 2)\text{-distributive}\}.$$

Corollary

Assume that $0^\sharp$ does not exist, and let $\mathbb{B}$ be a complete Boolean algebra and $2^{<\mathfrak{h}_2(\mathbb{B})} = \mathfrak{h}_2(\mathbb{B})$. Then White has a winning strategy in $\mathcal{G}_{\mathrm{ls}}(\mathfrak{h}_2(\mathbb{B}))$ iff forcing by $\mathbb{B}$ collapses $2^{\mathfrak{h}_2(\mathbb{B})}$ to $\mathfrak{h}_2(\mathbb{B})$ in some generic extension.

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Playing on a singular cardinal

Theorem

If White has a winning strategy in the game $\mathcal{G}_{ls}(\kappa)$ played on $\mathbb{B}$, then White has a winning strategy in $\mathcal{G}_{ls}(\mathrm{cf}(\kappa))$ as well.

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If Black has a winning strategy in the game $\mathcal{G}_{ls}(\mathrm{cf}(\kappa))$ played on $\mathbb{B}$, then Black has a winning strategy in $\mathcal{G}_{ls}(\kappa)$ as well.

The converse of neither of two theorems is true (an example: $\mathrm{Col}(\aleph_0, \aleph_1)$ under CH).

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Prescribing when a player has a winning strategy

Theorem

(GCH) For each set S of regular cardinals there is a complete Boolean algebra $\mathbb{B}$ such that
(a) $\text{White}(\mathbb{B}) = S$;
(b) $\text{Black}(\mathbb{B}) = \text{Card} \setminus (S \cup \omega)$.

A Boolean algebra on which the game is undetermined

If S is a stationary subset of κ:
$\diamondsuit_\kappa(S)$: There are sets $A_\gamma \subseteq \gamma$ for $\gamma \in S$ such that for each $A \subseteq \kappa$ the set $\{\gamma \in S : A \cap \gamma = A_\gamma\}$ is a stationary subset of κ.
$E(\kappa)$-the (stationary) set of all ordinals $< \kappa^+$ of cofinality κ.

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For each regular κ satisfying $\kappa^{<\kappa} = \kappa$ and $\diamondsuit_{\kappa^+}(E(\kappa))$, there is a κ^+-Suslin tree $\langle T, \leq \rangle$ such that the game $\mathcal{G}_{\mathrm{ls}}(\kappa)$ is undetermined on the algebra $\mathbb{B} = \mathrm{r.o.}(\langle T, \geq \rangle)$.

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