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1. What are large cardinals?

2. Forcings which preserve large cardinals (failure of GCH at a measurable)

3. Forcings which destroy large cardinals, but do something interesting (Singular Cardinal Hypothesis)

4. Some open questions

What are large cardinals?

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κ is *measurable* iff:
$\kappa > \aleph_0$
$\exists$ nonprincipal, κ-complete ultrafilter on κ

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Idea: κ is "large" iff κ is the critical point of an embedding
$j : V \to M$ where M is "large"

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However: κ could be λ-hypermeasurable for all λ (i.e., the critical point of embeddings with arbitrary degrees of hypermeasurability)

Forcings that preserve large cardinals

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If G^* belongs to $V[G]$ then κ is still measurable (and maybe more) in $V[G]$

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An example: Making GCH fail at a measurable cardinal

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Adds κ^{++}-many κ-Cohen sets

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Better choice: Sacks(κ, κ^{++})
Adds κ^{++}-many κ-Sacks subsets of κ (defined later)

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Solution: Force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

where $P(\alpha)$ denotes α-Cohen forcing.
Let $C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the P-generic

Forcings that preserve large cardinals

Now we want to lift $j : V \to M$ to
$j^* : V[C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)] \to M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$
where the β_i's are the inaccessibles of M between κ and $j(\kappa)$.

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To find the C^*'s:

Set $C^*(\alpha) = C(\alpha)$ for $\alpha < \kappa$

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Last lift: Take $C^*(j(\kappa))$ to be any generic for $j(\kappa)$-Cohen forcing of $M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$ containing the condition $C(\kappa) = C^*(\kappa)$ (such generics exist).

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For inaccessible $\alpha \leq \kappa$ replace α-Cohen by Cohen(α, α^{++})

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For inaccessible $\alpha \leq \kappa$ replace α-Cohen by Cohen(α, α^{++})
All goes well until the last lift: we *can* choose $C^*(\gamma)$ for all M-inaccessible $\gamma < j(\kappa)$ and lift $j : V \to M$ to
$j' : V[C(\alpha_0) * C(\alpha_1) * \cdots] \to$
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But Cohen$(j(\kappa), j(\kappa^{++}))$ is a very big forcing (it may have no generic; we may have to force one!) and $j'[C(\kappa)]$ is a very complicated set of conditions in this forcing (it is not easy to force a generic that contains it!)

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Lemma

(Tuning Fork Lemma) Suppose that $j : V \to M$ has critical point κ and g is κ-Sacks generic. Then in $V[g]$ there are exactly two generics h_0, h_1 for the $j(\kappa)$-Sacks of M extending g; moreover $h_0(\kappa) = 0$ and $h_1(\kappa) = 1$.

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A similar result holds for Sacks(κ, κ^{++}), thereby solving the problem of the "last lift".

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(with Dobrinen) Assume GCH and let κ be κ^{++}-hypermeasurable. Then there is a forcing extension in which κ is still measurable and the tree property holds at κ^{++}.

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(with Zdomskyy) Assume GCH and let κ be κ^{++}-hypermeasurable. Then there is a cofinality-preserving forcing extension in which κ is still measurable and the symmetric group on κ has cofinality κ^{++}.

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Theorem

(Prikry) Suppose that κ is measurable and the GCH fails at κ. Then in a forcing extension, κ is still a strong limit cardinal where the GCH fails, but now κ has cofinality ω. In particular, the SCH fails in this forcing extension.

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Prikry forcing: A forcing that preserves cardinals, adds no new bounded subsets of κ but adds an ω-sequence cofinal in κ

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Conditions in Prikry forcing:

Fix a normal measure U on κ. A condition is a pair (s, A) where s is a finite subset of κ and A belongs to U.

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(b) P is κ^+-cc: If $(s, A), (t, B)$ are conditions and $s = t$ then (s, A) and (t, B) are compatible.

Forcings which use large cardinals: The SCH

The main lemma about Prikry forcing is the following. We say that (t, B) is a *direct extension* of (s, A) iff $s = t$ and B is a subset of A.

Lemma (The Prikry property)

For σ a sentence of the forcing language, every condition has a direct extension which decides σ (i.e., either forces σ or $\sim \sigma$).

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Proof. Suppose that (s, A) is a condition and define $h : [A]^{<\omega} \longrightarrow 2$ as follows:

$h(t) = 1$ iff $(s \cup t, B) \Vdash \sigma$ for some B
$h(t) = 0$ otherwise.

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As U is normal there is $A^* \in U$ which is *homogeneous* for h: For each n and t_1, $t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$.

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As U is normal there is $A^* \in U$ which is *homogeneous* for h: For each n and t_1, $t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$. Then (s, A^*) decides σ: Otherwise there would be $(s \cup t_1, B_1)$, $(s \cup t_2, B_2)$ extending (s, A^*) which force σ, $\sim \sigma$, respectively. We can assume that for some n, both t_1 and t_2 belong to $[A^*]^n$. But then $h(t_1) = 0$, $h(t_2) = 1$, contradicting homogeneity. $\square$

Forcings which use large cardinals: The SCH

Corollary: P does not add new bounded subsets of κ.

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In summary: If G is P-generic then κ has cofinality ω in $V[G]$ and V, $V[G]$ have the same cardinals and bounded subsets of κ. In particular, if GCH fails at κ in V, then in $V[G]$, κ is a singular strong limit cardinal where the GCH fails.

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For the proof, mix Prikry forcing with *Lévy collapses*:

Suppose that $\alpha < \beta$ are regular. Then Lévy(α, β) is a forcing that makes β into α^+ and otherwise preserves cardinals:

$p \in$ Lévy(α, β) iff p is partial function of size $< \alpha$ from $\alpha \times \beta$ to β such that $p(\alpha_0, \beta_0) < \beta_0$ for each (α_0, β_0) in the domain of p.

Forcings which use large cardinals: The SCH

Collapsing Prikry forcing: 1st try

Fix a normal measure U on κ. A condition is of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible
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Problem: This collapses κ to ω (the p_i's are running wild!)

Solution: Control the p_i's on a measure one set

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The forcing is κ^+-cc. But why isn't κ collapsed?

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Then Prikry Collapse forcing makes κ into $\aleph_\omega$ with $\aleph_\omega$ strong limit, GCH failing at $\aleph_\omega$ (Strong failure of the SCH)

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Is it consistent with a supercompact cardinal for $H(\kappa^+)$ to have a definable wellordering for every uncountable κ?

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