Forcing when there are Large Cardinals

Summary:
1. What are large cardinals?
2. Forcings which preserve large cardinals (failure of GCH at a measurable)
3. Forcings which destroy large cardinals, but do something interesting (Singularity Cardinal Hypothesis)
4. Some open questions
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2. Forcings which preserve large cardinals (failure of GCH at a measurable)

3. Forcings which destroy large cardinals, but do something interesting (Singular Cardinal Hypothesis)

4. Some open questions
What are large cardinals?

\( \kappa \) is inaccessible iff:
\( \kappa > \aleph_0 \)
\( \kappa \) is regular
\( \lambda < \kappa \rightarrow 2^\lambda < \kappa \)
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\( \kappa \) is measurable iff:
- \( \kappa > \aleph_0 \)
- \( \exists \) nonprincipal, \( \kappa \)-complete ultrafilter on \( \kappa \)
What are large cardinals?

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- \( j \) is not the identity
- \( j \) preserves the truth of formulas with parameters

*Critical point* of \( j \) is the least \( \kappa \), \( j(\kappa) \neq \kappa \)

Idea: \( \kappa \) is “large” iff \( \kappa \) is the critical point of an embedding \( j : V \rightarrow M \) where \( M \) is “large”
What are large cardinals?

Suppose that $\kappa$ is the critical point of $j : V \rightarrow M$. 

Fact: Measurable $= \kappa$-hypermeasurable $= \kappa$-supercompact.

Kunen: No $j : V \rightarrow M$ witnesses $\lambda$-hypermeasurability for all $\lambda$, i.e., $M$ cannot equal $V$.

However: $\kappa$ could be $\lambda$-hypermeasurable for all $\lambda$ (i.e., the critical point of embeddings with arbitrary degrees of hypermeasurability)
What are large cardinals?

Suppose that $\kappa$ is the critical point of $j : V \rightarrow M$

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Forcings that preserve large cardinals

Question: Suppose $\kappa$ is a large cardinal and $G$ is $P$-generic over $V$. Is $\kappa$ still a large cardinal in $V[G]$?
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Goal: Find a \( G^* \) which is \( P^* \)-generic over \( M \) such that \( j[G] \subseteq G^* \)
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Then $j : V \rightarrow M$ lifts to $j^* : V[G] \rightarrow M[G^*]$, defined by $j^*(\sigma^G) = j(\sigma)^{G^*}$
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If $G^*$ belongs to $V[G]$ then $\kappa$ is still measurable (and maybe more) in $V[G]$. 
Forcings that preserve large cardinals

An example: Making GCH fail at a measurable cardinal

**Theorem**

Suppose that \( \kappa \) is \( \kappa^{++} \)-hypermeasurable. Then in a forcing extension, \( \kappa \) is still measurable and \( 2^\kappa = \kappa^{++} \).
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Obvious choice: Cohen($\kappa$, $\kappa^{++}$)

Add $\kappa^{++}$-many $\kappa$-Cohen sets
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Better choice: Sacks$(\kappa, \kappa^{++})$

Adds $\kappa^{++}$-many $\kappa$-Sacks subsets of $\kappa$ (defined later)
Forcings that preserve large cardinals

Step 2: Prepare below $\kappa$

Here is the problem (illustrated using just $\kappa$-Cohen forcing):

Suppose that $C \subseteq \kappa$ is $\kappa$-Cohen generic.

Want to lift $j: V \to M$ to $j^*: V[C] \to M[C^*]$.

Need to find $C^*$ which is $\kappa(\kappa)$-Cohen generic over $M$ and extends $C$, i.e., such that $C = C^* \cap \kappa$.

Impossible! $C$ does not belong to $M$!

Need the forcing to lift $C^*$ to be defined not in $M$ but in a model that already has $C$.

Solution: Force not just at $\kappa$, but at all inaccessible $\alpha \leq \kappa$, via an iteration $P = P(\alpha_0) \ast P(\alpha_1) \ast \cdots \ast P(\kappa)$ where $P(\alpha)$ denotes $\alpha$-Cohen forcing.
Forcings that preserve large cardinals

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$$P = P(\alpha_0) \ast P(\alpha_1) \ast \cdots \ast P(\kappa)$$

where $P(\alpha)$ denotes $\alpha$-Cohen forcing.
Let $C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)$ denote the $P$-generic
Forcings that preserve large cardinals

Now we want to lift \( j : V \to M \) to

\[
j^* : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)] \to M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \ast C^*(j(\kappa))]\]

where the \( \beta_i \)'s are the inaccessibles of \( M \) between \( \kappa \) and \( j(\kappa) \).
Forcings that preserve large cardinals

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$$j^* : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)] \rightarrow$$

$$M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \ast C^*(j(\kappa))]$$

where the $\beta_i$’s are the inaccessibles of $M$ between $\kappa$ and $j(\kappa)$.

To find the $C^*$’s:

Set $C^*(\alpha) = C(\alpha)$ for $\alpha < \kappa$
Forcings that preserve large cardinals

Now we want to lift $j : V \rightarrow M$ to

$$j^* : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)] \rightarrow \ M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \ast C^*(j(\kappa))]$$

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Now we want to lift \( j : V \rightarrow M \) to
\[
\begin{align*}
  j^* : V[C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)] & \rightarrow \\
  M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))] & \\
\end{align*}
\]
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Take \( \langle C^*(\beta) \mid \kappa < \beta < j(\kappa) \rangle \) to be any generic (they exist)
Forcings that preserve large cardinals

Now we want to lift $j : V \to M$ to

$$j^* : V[C(\alpha_0) \ast C(\alpha_1) \ast \cdots \ast C(\kappa)] \to M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots \ast C^*(j(\kappa))]$$

where the $\beta_i$’s are the inaccessibles of $M$ between $\kappa$ and $j(\kappa)$.

To find the $C^*$’s:

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Set $C^*(\kappa) = C(\kappa)$

Take $\langle C^*(\beta) \mid \kappa < \beta < j(\kappa) \rangle$ to be any generic (they exist)

Last lift: Take $C^*(j(\kappa))$ to be any generic for $j(\kappa)$-Cohen forcing of

$$M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots]$$

containing the condition $C(\kappa) = C^*(\kappa)$ (such generics exist).
Forcings that preserve large cardinals

Step 3: Make this work with \( \kappa \)-Cohen forcing replaced by some forcing that kills the GCH at \( \kappa \)
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Here is the problem:

For inaccessible $\alpha \leq \kappa$ replace $\alpha$-Cohen by $\text{Cohen}(\alpha, \alpha^{++})$
Forcings that preserve large cardinals

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Here is the problem:

For inaccessible $\alpha \leq \kappa$ replace $\alpha$-Cohen by Cohen($\alpha, \alpha^{++}$)

All goes well until the last lift: we can choose $C^*(\gamma)$ for all $M$-inaccessible $\gamma < j(\kappa)$ and lift $j : V \rightarrow M$ to $j' : V[C(\alpha_0) * C(\alpha_1) * \cdots] \rightarrow M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$
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We then need to find a generic for the Cohen($j(\kappa), j(\kappa^{++})$)-forcing of $M[C^*(\alpha_0) \ast C^*(\alpha_1) \ast \cdots \ast C^*(\kappa) \ast C^*(\beta_0) \ast C^*(\beta_1) \ast \cdots]$ which contains $j'[C(\kappa)]$. 
Forcings that preserve large cardinals

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which contains $j'[C(\kappa)]$.

But Cohen$(j(\kappa), j(\kappa^{++}))$ is a very big forcing (it may have no generic; we may have to force one!) and $j'[C(\kappa)]$ is a very complicated set of conditions in this forcing (it is not easy to force a generic that contains it!)
Forcings that preserve large cardinals

Here is the solution: Use Sacks($\kappa, \kappa^{++}$) instead of Cohen($\kappa, \kappa^{++}$)
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Here is the solution: Use Sacks$(\kappa, \kappa^{++})$ instead of Cohen$(\kappa, \kappa^{++})$
Then we don’t have to build a generic $S^*(j(\kappa))$ for
Sacks$(j(\kappa), j(\kappa^{++}))$ because $j'[S(\kappa)]$ builds one for us!
Forcings that preserve large cardinals

Here is the solution: Use $\text{Sacks}(\kappa, \kappa^{++})$ instead of $\text{Cohen}(\kappa, \kappa^{++})$. Then we don’t have to build a generic $S^*(j(\kappa))$ for $\text{Sacks}(j(\kappa), j(\kappa^{++}))$ because $j'[S(\kappa)]$ builds one for us!

Illustrate with $\kappa$-Sacks: A condition is a perfect $\kappa$-tree with a closed unbounded set of splitting levels.
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Illustrate with $\kappa$-Sacks: A condition is a perfect $\kappa$-tree with a closed unbounded set of splitting levels. If $G$ is generic then the intersection of the $\kappa$-trees in $G$ gives us a function $g : \kappa \to 2$. 
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unbounded set of splitting levels. If $G$ is generic then the
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Lemma

(Tuning Fork Lemma) Suppose that $j : V \to M$ has critical point $\kappa$
and $g$ is $\kappa$-Sacks generic. Then in $V[g]$ there are exactly two
generics $h_0, h_1$ for the $j(\kappa)$-Sacks of $M$ extending $g$; moreover
$h_0(\kappa) = 0$ and $h_1(\kappa) = 1$. 
Forcings that preserve large cardinals

Here is the solution: Use Sacks\((\kappa, \kappa^{++})\) instead of Cohen\((\kappa, \kappa^{++})\)
Then we don’t have to build a generic \(S^*(j(\kappa))\) for
Sacks\((j(\kappa), j(\kappa^{++}))\) because \(j'[S(\kappa)]\) builds one for us!

Illustrate with \(\kappa\)-Sacks: A condition is a perfect \(\kappa\)-tree with a closed
unbounded set of splitting levels. If \(G\) is generic then the
intersection of the \(\kappa\)-trees in \(G\) gives us a function \(g : \kappa \to 2\).

Lemma

*(Tuning Fork Lemma)* Suppose that \(j : V \to M\) has critical point \(\kappa\)
and \(g\) is \(\kappa\)-Sacks generic. Then in \(V[g]\) there are exactly two
generics \(h_0, h_1\) for the \(j(\kappa)\)-Sacks of \(M\) extending \(g\); moreover
\(h_0(\kappa) = 0\) and \(h_1(\kappa) = 1\).

A similar result holds for Sacks\((\kappa, \kappa^{++})\), thereby solving the
problem of the “last lift”.
Forcings that preserve large cardinals

Some other applications:
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(with Magidor) Assume GCH, let $\kappa$ be measurable and let $\alpha$ be any cardinal at most $\kappa^{++}$. Then there is a cofinality-preserving forcing extension in which there are exactly $\alpha$-many normal measures on $\kappa$. 
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(with Magidor) Assume GCH, let $\kappa$ be measurable and let $\alpha$ be any cardinal at most $\kappa^{++}$. Then there is a cofinality-preserving forcing extension in which there are exactly $\alpha$-many normal measures on $\kappa$.

(with Dobrinen) Assume GCH and let $\kappa$ be $\kappa^{++}$-hypermeasurable. Then there is a forcing extension in which $\kappa$ is still measurable and the tree property holds at $\kappa^{++}$.
Forcings that preserve large cardinals

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(with Magidor) Assume GCH, let $\kappa$ be measurable and let $\alpha$ be any cardinal at most $\kappa^{++}$. Then there is a cofinality-preserving forcing extension in which there are exactly $\alpha$-many normal measures on $\kappa$.

(with Dobrinen) Assume GCH and let $\kappa$ be $\kappa^{++}$-hypermeasurable. Then there is a forcing extension in which $\kappa$ is still measurable and the tree property holds at $\kappa^{++}$.

(with Zdomskyy) Assume GCH and let $\kappa$ be $\kappa^{++}$-hypermeasurable. Then there is a cofinality-preserving forcing extension in which $\kappa$ is still measurable and the symmetric group on $\kappa$ has cofinality $\kappa^{++}$. 
Forcings which use large cardinals: The SCH

The Singular cardinal hypothesis (SCH):

If $2^{\text{cof}(\kappa)} < \kappa$ then $\text{cof}(\kappa) = \kappa +$. SCH $\Rightarrow$ GCH holds at singular strong limit cardinals

Theorem (Prikry) Suppose that $\kappa$ is measurable and the GCH fails at $\kappa$. Then in a forcing extension, $\kappa$ is still a strong limit cardinal where the GCH fails, but now $\kappa$ has cofinality $\omega$. In particular, the SCH fails in this forcing extension.

Prikry forcing: A forcing that preserves cardinals, adds no new bounded subsets of $\kappa$ but adds an $\omega$-sequence cofinal in $\kappa$. 
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**Theorem**

(Prikry) Suppose that $\kappa$ is measurable and the GCH fails at $\kappa$. Then in a forcing extension, $\kappa$ is still a strong limit cardinal where the GCH fails, but now $\kappa$ has cofinality $\omega$. In particular, the SCH fails in this forcing extension.
Forcings which use large cardinals: The SCH

Singular cardinal hypothesis (SCH):
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Prikry forcing: A forcing that preserves cardinals, adds no new bounded subsets of \(\kappa\) but adds an \(\omega\)-sequence cofinal in \(\kappa\).
Forcings which use large cardinals: The SCH

Conditions in Prikry forcing:

Fix a normal measure $U$ on $\kappa$. A condition is a pair $(s, A)$ where $s$ is a finite subset of $\kappa$ and $A$ belongs to $U$. 
Forcings which use large cardinals: The SCH

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Facts: (a) If $G$ is $P$-generic then $\bigcup\{s \mid (s, A) \in G \text{ for some } A\}$ is an $\omega$-sequence cofinal in $\kappa$.
(b) $P$ is $\kappa^+$-cc: If $(s, A), (t, B)$ are conditions and $s = t$ then $(s, A)$ and $(t, B)$ are compatible.
The main lemma about Prikry forcing is the following. We say that \((t, B)\) is a direct extension of \((s, A)\) iff \(s = t\) and \(B\) is a subset of \(A\).

**Lemma (The Prikry property)**

*For \(\sigma\) a sentence of the forcing language, every condition has a direct extension which decides \(\sigma\) (i.e., either forces \(\sigma\) or \(\sim \sigma\)).*
### Forcings which use large cardinals: The SCH

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Forcings which use large cardinals: The SCH

Lemma (The Prikry property)

For $\sigma$ a sentence of the forcing language, every condition has a direct extension which decides $\sigma$ (i.e., either forces $\sigma$ or $\sim \sigma$).

Proof. Suppose that $(s, A)$ is a condition and define $h : [A]^{<\omega} \to 2$ as follows:

$h(t) = 1$ iff $(s \cup t, B) \models \sigma$ for some $B$
$h(t) = 0$ otherwise.
For forcings which use large cardinals: The SCH

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As $U$ is normal there is $A^* \in U$ which is homogeneous for $h$: For each $n$ and $t_1, t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$. 
Forcings which use large cardinals: The SCH

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As $U$ is normal there is $A^* \in U$ which is homogeneous for $h$: For each $n$ and $t_1, t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$. Then $(s, A^*)$ decides $\sigma$: Otherwise there would be $(s \cup t_1, B_1), (s \cup t_2, B_2)$ extending $(s, A^*)$ which force $\sigma, \sim \sigma$, respectively. We can assume that for some $n$, both $t_1$ and $t_2$ belong to $[A^*]^n$. But then $h(t_1) = 0, h(t_2) = 1$, contradicting homogeneity. $\square$
Corollary: $P$ does not add new bounded subsets of $\kappa$.

Proof. Suppose $(s, A) \models \dot{a}$ is a subset of $\lambda$, where $\lambda$ is less than $\kappa$. Set $(s, A_0) = (s, A)$ and using the Prirky property choose a direct extension $(s, A_1)$ of $(s, A_0)$ which decides “$0 \in \dot{a}$”. Then choose a direct extension $(s, A_2)$ of $(s, A_1)$ which decides “$1 \in \dot{a}$”, etc. After $\lambda$ steps we have a direct extension $(s, A_\lambda)$ of $(s, A)$ which decides which ordinals less than $\lambda$ belong to $\dot{a}$, and therefore forces $\dot{a}$ to belong to the ground model. □
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In summary: If $G$ is $P$-generic then $\kappa$ has cofinality $\omega$ in $V[G]$ and $V, V[G]$ have the same cardinals and bounded subsets of $\kappa$. In particular, if GCH fails at $\kappa$ in $V$, then in $V[G]$, $\kappa$ is a singular strong limit cardinal where the GCH fails.
Forcings which use large cardinals: The SCH

An improvement: Model where ℵ_ω is strong limit and the GCH fails at ℵ_ω.
Forcings which use large cardinals: The SCH

An improvement: Model where $\aleph_\omega$ is strong limit and the GCH fails at $\aleph_\omega$

**Theorem**

*(Magidor)* Suppose that $\kappa$ is measurable. Then there is a forcing extension in which $\kappa$ equals $\aleph_\omega$. 

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For the proof, mix Prikry forcing with Lévy collapses:
Forcings which use large cardinals: The SCH

An improvement: Model where $\aleph_\omega$ is strong limit and the GCH fails at $\aleph_\omega$

**Theorem**

(Magidor) Suppose that $\kappa$ is measurable. Then there is a forcing extension in which $\kappa$ equals $\aleph_\omega$.

For the proof, mix Prikry forcing with Lévy collapses:

Suppose that $\alpha < \beta$ are regular. Then Lévy($\alpha, \beta$) is a forcing that makes $\beta$ into $\alpha^+$ and otherwise preserves cardinals:

$p \in \text{Lévy}(\alpha, \beta)$ iff $p$ is partial function of size $< \alpha$ from $\alpha \times \beta$ to $\beta$ such that $p(\alpha_0, \beta_0) < \beta_0$ for each $(\alpha_0, \beta_0)$ in the domain of $p$. 
Collapsing Prikry forcing: 1st try

Fix a normal measure $U$ on $\kappa$. A condition is of the form

$((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A)$

where:

- $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible
- $p_i$ belongs to $\text{Lévy}(\alpha_i, \alpha_{i+1})$ for $i < n - 1$
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To extend: Strengthen the $p_i$’s, increase $n$, shrink $A$ and take the new $\alpha$’s from the old $A$
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Problem: This collapses $\kappa$ to $\omega$ (the $p_i$’s are running wild!)

Solution: Control the $p_i$’s on a measure one set
Collapsing Prikry forcing: 2nd try
Let $j : V \rightarrow M$ witness that $\kappa$ is measurable and choose $U$ to be the normal measure $\{A \mid \kappa \in j(A)\}$
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Forcings which use large cardinals: The SCH

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- \( j(F)(\kappa) \) belongs to \( G \)
Forcings which use large cardinals: The SCH

An extension of
\[ p = ((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F) \]
is of the form
\[ p^* = ((\alpha_0^*, p_0^*), (\alpha_1^*, p_1^*), \ldots, (\alpha_{n^*-1}^*, p_{n^*-1}^*), A^*, F^*) \]
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A generic produces a Prikry sequence \( \alpha_0 < \alpha_1 < \cdots \) in \( \kappa \) together with Lévy collapses \( g_0, g_1, \ldots \) where \( g_i \) ensures \( \alpha_{i+1} = \alpha_i^{++} \). So after collapsing \( \alpha_0 \), we see that \( \kappa \) is at most \( \aleph_\omega \).
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Forcings which use large cardinals: The SCH

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The forcing is \( \kappa^+-cc \). But why isn’t \( \kappa \) collapsed?
Forcings which use large cardinals: The SCH

The Prikry property: For $\sigma$ a sentence of the forcing language, every condition has a direct extension which decides $\sigma$. 
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Using this, one gets: Any bounded subset of $\kappa$ belongs to $V[g_0, g_1, \ldots, g_n]$ for some $n$, and therefore $\kappa$ remains a cardinal.
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Using this, one gets: Any bounded subset of $\kappa$ belongs to $V[g_0, g_1, \ldots, g_n]$ for some $n$, and therefore $\kappa$ remains a cardinal.

Summary: Prikry Collapse forcing makes $\kappa$ into $\aleph_\omega$ and preserves cardinals above $\kappa$. 

Now start with $\kappa$ measurable and GCH failing at $\kappa$.
The Prikry property: For $\sigma$ a sentence of the forcing language, every condition has a direct extension which decides $\sigma$.

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Then Prikry Collapse forcing makes $\kappa$ into $\aleph_\omega$ with $\aleph_\omega$ strong limit, GCH failing at $\aleph_\omega$ (Strong failure of the SCH)
Open Questions

1. Preserving large cardinals

Consider various cardinal characteristics of the continuum (almost-disjointness number, bounding number, dominating number, splitting number, ...)

How do these behave at a large cardinal?

Is it consistent that a strongly compact cardinal have a unique normal measure?

Is it consistent with a supercompact cardinal for $H(\kappa^+)$ to have a denable well-ordering for every uncountable $\kappa$?
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Can the nonstationary ideal on \( \omega_1 \) be saturated with CH?
Can \( \mathfrak{N}_\omega \) be Jonsson?