FORCING AND RAMSEY THEOREMS

Jindřich Zapletal

University of Florida
Czech Academy of Sciences

Basic set-up for definable forcing

For a Polish space X and a σ-ideal I, P_I denotes the partial order of Borel I-positive sets ordered by inclusion.

Fact. Most "definable" proper forcings P adding a single element $\dot{x}$ of the space X are equivalent to P_I where $I = \{B \subset X : B$ is Borel and $P \Vdash \dot{x} \notin \dot{B}$.

Examples. Random forcing and the ideal of measure zero sets. Sacks forcing and the ideal of countable sets.

Basic features

Suppose X is a Polish space and I is a σ-ideal on it such that P_I is proper. Then

- the poset P_I adds a point $\dot{x} \in X$ which belongs to all sets in the generic filter;

- for every name $\dot{y}$ for a point in a Polish space Y there is a Borel function $f : b \to Y$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x})$;

- for every name $\dot{C}$ for a Borel subset of Y there is a Borel set $D \subset B \times C$ such that $B \Vdash \dot{C} = \dot{D}_{\dot{x}}$.

Not adding a splitting real

A *splitting real* over a model M of ZFC is a set $a \subset \omega$ with an infinite intersection with every infinite set in M.

Fact. Suppose X is a Polish space and I is a σ-ideal on it and the forcing P_I is proper. The following are equivalent:

- P_I adds no splitting real
- For every I-positive Borel set $B \subset X$ and every Borel partition $B \times \omega = D_0 \cup D_1$, one piece of the partition contains a set of the form Borel I-positive$\times$ infinite.

Example. Let I be the σ-ideal on $[0,1]$ generated by sets of finite 1/2-dimensional Hausdorff measure. Then $I^+ \times \omega \rightarrow_B I^+ \times$infinite.

Question. Let J be the σ-ideal on $\mathbb{R}^3$ consisting of sets of zero Newtonian capacity. Is it true that $J^+ \times \omega \rightarrow_B J^+ \times$infinite?

Similar. Given Borel sets $B_n : n \in \omega$ of capacity $< \varepsilon$, can you find two of them whose union has capacity $< \varepsilon + \delta$?

Strong preservation of measure

A poset *strongly preserves Lebesgue measure* if every closed set in the extension has a closed subset from the ground model with arbitrarily close measure.

Fact. Suppose that X is a Polish space and I is a σ-ideal on it and the forcing P_I is proper. The following are equivalent:

- P_I strongly preserves measure
- for every Borel I-positive set $B \subset X$ and every Borel partition $B \times [0,1] = \bigcup_n D_n$, one of the pieces of the partition contains a rectangle of the form Borel I-positive$\times$ Borel non-null.

Strong preservation of category

A poset *strongly preserves Baire category* if every Borel nonmeager set in the extension has a Borel nonmeager subset from the ground model.

Fact. Suppose that X is a Polish space and I is a σ-ideal on it and the forcing P_I is proper. The following are equivalent:

- P_I strongly preserves category
- for every Borel I-positive set $B \subset X$ and every Borel partition $B \times [0,1] = \bigcup_n D_n$, one of the pieces of the partition contains a rectangle of the form Borel I-positive$\times$ Borel non-meager.

Product forcing and rectangular Ramsey theorems

Question. If X, I and Y, J are Polish spaces and σ-ideal and the quotient forcings P_I, P_J are proper, the the product may (or may not?) be proper. Can we calculate the ideal K on $X \times Y$ such that $P_I \times P_J$ is equivalent to P_K?

Definition. The ideals I, J have *rectangular Ramsey property* if for every Borel partition of $B \times C$ into countably many pieces, one of the pieces contains a Borel rectangle with I-positive and J-positive sides.

Observation. If the ideals I, J have the property then the collection of Borel set without a positive Borel rectangular subset form a σ-ideal K and $P_I \times P_J$ is densely embedded in P_K.

Theorem. If I, J are suitably definable ideals such that the forcings P_I, P_J are proper and strongly preserve Baire category, then I, J have the rectangular Ramsey property and the forcing $P_I \times P_J$ is proper and strongly preserves Baire category.

Remark. The same theorem holds for countable products. It seems optimal. Suppose that the product of countably many definable forcings is proper. Does it imply that all but finitely many of them strongly preserve Baire category?

Forcing and canonization theorems

Definition. An equivalence E on a Polish space X is smooth if there is a Borel function $f : X \to 2^\omega$ such that $xEy \leftrightarrow f(x) = f(y)$.

Fact. Suppose that I is a σ-ideal on a space X such that the quotient P_I is proper. The following are equivalent:

- P_I adds a minimal real;
- every smooth equivalence relation on a Borel positive set is equal to identity or everything on a smaller Borel positive set.

Question. How about non-smooth equivalence relations?

Case 1. Sacks forcing. The associated ideal is the ideal of countable sets on 2^ω.

Fact. (Silver) Every Borel equivalence relation on 2^ω either has countably many classes or it contains a perfect set of pairwise incompatible elements.

Conclusion. Any Borel equivalence relation on a perfect set is equal to everything or to the identity on a perfect subset. In fact, this is true for any suitably definable equivalence relation.

Case 2. Silver forcing. The conditions are functions from ω to 2 with co-infinite domain, ordered by reverse inclusion. The associated ideal is σ-generated by all Borel sets $B \subset 2^\omega$ such that every two distinct elements of B differ at more than just one entry.

Observation. The E_0 equivalence (equality modulo finite on 2^ω) does not equal to identity or everything on any positive Borel set.

Theorem. Every Borel equivalence is equal to either everything or to the identity or to E_0 on a positive set.

Forcings directly derived from Ramsey theorems

Fact. Milliken's theorem. For every partition of finite subsets of ω into two classes there is an infinite collection of disjoint sets such that the union of any finite subcollection of them falls into the same piece of the partition.

Derived forcing. A condition $p = \langle a_p, b_p \rangle$ where a_p is a finite set and b_p is an infinite collection of disjoint finite sets. $q \leq p$ if $a_q = a_p \cup$ finite subcollection of b_p and every set in b_q is union of sets in b_p.

The point. The properties of Milliken forcing depend on Milliken's theorem.

Theorem. The following items hold about the Milliken forcing P:

- P is proper
- P preserves Baire category and outer measure;
- P does not add a splitting real;
- P adds a minimal real.