Extremally Disconnected Topological Groups

Osvaldo A. Téllez Nieto
(joint work with Michael Hrušák)

Instituto de Matemáticas UNAM, Unidad Morelia

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Contents

Extremally disconnected topological spaces

Definition

A topological space (X, τ) is *extremally disconnected* (E.D.) iff for each open set A, $\bar{A}$ is open.

Equivalently, a topological space X is E.D. iff for every pair of open sets A, B, if $A \cap B = \emptyset$, then $\bar{A} \cap \bar{B} = \emptyset$.

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Question (Arkhangel'skii, 1968)

Is there a non-discrete extremally disconnected topological group?

A partial answer is: Consistently, yes. That is, CON(ZFC) implies CON(ZFC + There is a non-discrete E.D. topological group).

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Extremally Disconnected Topology

Consider $[\omega]^{<\omega}$, the set of finite subsets of natural numbers. We define two topologies on $[\omega]^{<\omega}$.

Definition

Let $\mathcal{F}$ be a filter on ω. We define $\tau_{\mathcal{F}}$ as follows: for each $U \subseteq [\omega]^{<\omega}$, $U \in \tau_{\mathcal{F}}$ iff for all $X \in U$, $\{n \in \omega : X \cup \{n\} \in U\} \in \mathcal{F}$.

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Extremally Disconnected Topology

Proposition

If $\mathcal{F}$ is ultrafilter, then $\tau_{\mathcal{F}}$ is extremally disconnected.

Group topology

Now we define another topology $\tau^{\mathcal{F}}$ on $[\omega]^{<\omega}$.

Definition

Let $\mathcal{F}$ be an ultrafilter on ω and $X \in [\omega]^{<\omega}$. The basic $\tau^{\mathcal{F}}$-nhoods for X are the sets $A_X = \{X \triangle M : M \in [A]^{<\omega}\}$ where $A \in \mathcal{F}$ and $\triangle$ denotes the symmetric difference of sets.

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Group topology

Proposition

$[\omega]^{<\omega}$ with the topology $\tau^{\mathcal{F}}$ is a topological group with $\triangle$ as operation.

Selective ultrafilter

Definition

Let $\mathcal{F}$ be an ultrafilter on ω. $\mathcal{F}$ is a *selective ultrafilter* iff for every function $\varphi : [\omega]^2 \to \{0, 1\}$, there exists $A \in \mathcal{F}$ homogeneous, that is, $\varphi''([A]^2) = i$ for some $i \in \{0, 1\}$.

Louveau's Theorem

Theorem (Louveau, 1972)

The following statements are equivalent:

1. $\mathcal{F}$ *is a selective ultrafilter*
2. $\tau^{\mathcal{F}} = \tau_{\mathcal{F}}$
3. $([\omega]^{<\omega}, \tau_{\mathcal{F}})$ *is a topological group.*

Louveau's Theorem

The following statement is also equivalent to the previous statements:

4. $\tau^{\mathcal{F}}$ is an extremally disconnected topology.

Proof

(2) implies (4) immediatly. We only prove (4) implies (1). Suppose that $\tau^{\mathcal{F}}$ is an extremally disconnected topology and let $\varphi : [\omega]^2 \to \{0,1\}$ be a coloring. For each $n \in \omega$, let

$$A_0^n = \{m \in \omega : \varphi(\{n,m\}) = 0\}$$

and

$$A_1^n = \{m \in \omega : \varphi(\{n,m\}) = 1\}$$

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then, since $A_0^n \cup A_1^n \in \mathcal{F}$, for some (unique) $i \in \{0, 1\}$, $A_i^n \in \mathcal{F}$. (We can say that $\mathcal{F}$ "prefers" i for n).

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Then:

- $U \in \tau^{\mathcal{F}}$ because U is an union of open sets in $\tau^{\mathcal{F}}$.
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As $\tau^{\mathcal{F}}$ is extremally disconnected, $\bar{U}$ is open (and $0 \in \bar{U}$) then there is a basic nhood of 0. That is, there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq U$. The set A is homegeneous because if $n, m \in A$, then $\{n, m\} \in [A]^{<\omega}$ and consequently $\{n, m\} \in U$, then $n \in A_i$ and $m \in A_i^n$ or $m \in A_i$ and $n \in A_i^m$. In both cases, $\varphi(\{n, m\}) = i$. ■

Nowhere dense ultrafilters

The extremally disconnected topological group of the example, satisfies the following property:

Fact

For every continuous function $f : [\omega]^{<\omega} \to 2^\omega$ (where $[\omega]^{<\omega}$ has the topology $\tau_{\mathcal{F}}$ with $\mathcal{F}$ a selective ultrafilter) there exists an open set V such that $f[V]$ is nwd in 2^ω.

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Question (Hrušák)

Is it true that for every separable extremally disconnected topological group X and for every countinuous function $f : X \to 2^\omega$ there exists an open set V, such that $f[V]$ is nwd in 2^ω ?

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Proposition

If X is extremally disconnected, then the following are equivalent:

- $\mathbb{P} = RO(X)$ *doesn't add Cohen reals.*
- *For every continuous function* $f : X \to 2^\omega$ *there exists an open set* V *such that* $f[V]$ *is nwd in* 2^ω.

Recall the following:

Theorem (Blaszczyk-Shelah)

There exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real iff there exists a nwd ultrafilter on ω.

If the answer to the question of Hrušák is yes, from the Blaszczyk-Shelah's theorem and the previous proposition we would have the following:

Corollary ???

The existence of a separable non-discrete extremally disconnected topological group implies the existence of a nwd ultrafilter.

Thank you!