Examples concerning generic sets

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1. Generic sets in any forcing - the decision property
Outline

1. Generic sets in any forcing - the decision property
2. Generic sets in c.c.c. forcings - catching uncountable sets
Outline

1. Generic sets in any forcing - the decision property
2. Generic sets in c.c.c. forcings - catching uncountable sets
3. Generics in the Cohen forcing - composing functions with the generic function
Notation

1. For example, if $A$ is an atomless Boolean algebra, then $A^* = A \{0\}$ with the Boolean order.

2. The canonical name for the generic set $\dot{G} = \{<\dot{p}, p> : p \in \mathcal{P}\}$.

3. The canonical names for the ground model elements $\dot{x} = \{<\dot{y}, p> : y \in x, p \in \mathcal{P}\}$.

4. If $\mathcal{P} \parallel - \dot{\mathcal{P}}$ and $\mathcal{P} \parallel - (p, q \in \dot{\mathcal{G}} \Rightarrow \exists r \leq p, q [r \in \dot{\mathcal{G}}])$.

5. If $D \subseteq \mathcal{P}$ is dense, then $\mathcal{P} \parallel - \dot{\mathcal{D}} \cap \dot{\mathcal{G}} \neq \emptyset$.

6. If $A$ is an atomless complete Boolean algebra, then $[\phi] = \bigvee\{a \in A^* : a \parallel - \phi\}$.

7. If we can prove that $\mathcal{P} \parallel - \phi$, then we can prove that $\text{Con}(\text{ZFC})$ implies $\text{Con}(\text{ZFC} + \phi)$.
Notation

1. \( (P, \leq) \) partial order (nonatomic) with which we force,

\[ A^* = A \{0\} \]

with the Boolean order.

The canonical name for the generic set \( \dot{G} = \{ <\dot{p}, p> : p \in P \} \).

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\( p \parallel - \dot{p} \in \dot{G}, P \parallel - (p, q \in \dot{G} \Rightarrow \exists r \leq p, q \ [r \in \dot{G}] ) \)

If \( D \subseteq P \) is dense, then \( P \parallel - \dot{D} \cap \dot{G} \neq \emptyset \).

If \( A \) is an atomless complete Boolean algebra then \[ \left[ \phi \right] = \bigvee \{ a \in A^* : a \parallel - \phi \} \]

If we can prove that \( P \parallel - \phi \), then we can prove that Con(ZFC) implies Con(ZFC + \( \phi \)).
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Notation

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4. $p \Vdash \check{p} \in \dot{G}, \quad P \Vdash (p, q \in \dot{G} \Rightarrow \exists r \leq p, q \ [r \in \dot{G}])$

5. If $D \subseteq P$ is dense, then $P \Vdash \check{D} \cap \dot{G} \neq \emptyset$
(1) \((P, \leq)\) partial order (nonatomic) with which we force, for example if \(A\) is an atomless Boolean algebra, then \(A^* = A \setminus \{0\}\) with the Boolean order.

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(4) \(p \models \check{\exists} p \in \hat{G}, \quad P \models (p, q \in \hat{G} \Rightarrow \exists r \leq p, q [r \in \hat{G}])\).

(5) If \(D \subseteq P\) is dense, then \(P \models \check{\exists} D \cap \hat{G} \neq \emptyset\).

(6) If \(A\) is an atomless complete Boolean algebra then \([\phi] = \bigvee\{a \in A^*: a \models \check{\neg} \phi\}\).
Notation

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6. If $A$ is an atomless complete Boolean algebra then 
   $[\phi] = \bigvee \{a \in A^*: a \forces \phi\}$

7. If we can prove that $P \forces \phi$, then we can prove that $\text{Con}(\text{ZFC})$ implies $\text{Con}(\text{ZFC}+\phi)$
Example 1. The decision property
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Motivation:

Theorem

Suppose that $A$ is an atomless Boolean algebra. Then $A$ is an ultrafilter in $A$.

Proof.

For each $a \in A$ consider the dense set in $A$

$$D_a = \{ p \in A^*: p \leq a \text{ or } p \leq -a \}$$
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Motivation:

**Theorem**

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**Proof.**

For each $a \in A^*$ consider the dense set in $A^*$

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(The decision property)

For each formula $\phi$ the following set is dense in $P$:

$$\{ p \in P : p \models \neg \phi \text{ or } p \models \neg \neg \phi \}$$
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1. For each formula $\phi$ the following set is dense in $P$:

   \[\{ p \in P : p \models \neg \phi \text{ or } p \models \neg \neg \phi \}\]

2. $[\neg \phi] = \neg [\phi]$

3. $P \models [\neg \phi] \in \dot{G}$ or $[\phi] \in \dot{G}$
Problem

Using forcing and generic ultrafilters get nontrivial countably generated ultrafilters in uncountable Boolean algebras

\[ P \text{ is c.c.c. iff it does not have uncountable family of pairwise incompatible conditions} \]

If \( P \) is c.c.c and \( \alpha \) is a cardinal, then \( P \parallel^\ast \check{\alpha} \) is a cardinal, i.e., c.c.c. forcings preserve cardinals and in particular preserve the uncountable

i.e., if \( X \) is uncountable \( P \parallel^\ast \check{X} \) is uncountable

If \( P \) is countable, then it is c.c.c.
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3. i.e., if \( X \) is uncountable \( P \vDash \check{X} \) is uncountable
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3. i.e., if $X$ is uncountable $P \Vdash \check{X}$ is uncountable
4. If $P$ is countable, then it is c.c.c.
We say that $B \subseteq A$ is a dense subalgebra of $A$ iff for every $a \in A^*$ there is $b \in B^*$ such that $b \leq a$. 
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**Theorem**

Let $B$ be a dense subalgebra of $A$ then $B^* \parallel\!\!\parallel$ The filter of $A$ generated by $\mathring{G}_B$ is an ultrafilter of $A$
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**Theorem**

Let \( B \) be a dense subalgebra of \( A \) then \( B^* \) \( \Vdash \) The filter of \( A \) generated by \( \hat{G}_B \) is an ultrafilter of \( A \)

**Proof.**

For each \( a \in A^* \) consider the dense set in \( B^* \)

\[
D_a = \{ b \in B^* : b \leq a \text{ or } b \leq -a \}
\]
Definition

Let $B \subseteq A$ be Boolean algebras. We say that $B$ is deep in $A$ if and only if

$$\forall a \in A \quad \forall b \in B \quad \exists c \in B \quad c \leq b \cap a \quad \text{or} \quad c \leq b - a.$$
Definition
Let $B \subseteq A$ be Boolean algebras. We say that $B$ is deep in $A$ if and only if
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Theorem
Suppose that $B$ is a countable deep subalgebra of an algebra $A$. Then $B^* \models A$ has a countably generated ultrafilter.
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Theorem
It is consistent with arbitrary big continuum that each $A \subseteq \wp(N)$ of countable independence has an ultrafilter which is countably or $\omega_1$-generated.
Example 2. Catching uncountable sets
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Motivation:
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Motivation:

Theorem

Suppose that $P$ satisfies c.c.c., and $X \subseteq P$ is uncountable, then there is $p \in P$ such that $p \models \check{X} \cap \check{G}$ is uncountable

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Theorem

Suppose that $P$ satisfies c.c.c., and $X \subseteq P$ is uncountable, then there is $p \in P$ such that $p \models \check{X} \cap \dot{G}$ is uncountable.

Proof.

1. We may w.l.o.g. assume that $X = \{ x_\alpha : \alpha < \omega_1 \}$.
2. If the theorem is false, for each $p$ there is $q \leq p$ and $\alpha_q$ such that $q \models \check{X} \cap \dot{G} \subseteq \{ \check{x}_\alpha : \alpha < \check{\alpha}_q \} \& \check{X} \cap \dot{G} \not\subseteq \{ \check{x}_\alpha : \alpha < \check{\beta} \}$ for $\check{\beta} < \check{\alpha}_q$.
3. There cannot be uncountably many conditions which force pairwise contradictory information, so $\{ \alpha_q : q \in P \}$ is countable, so it has its supremum $\beta < \omega_1$ which satisfies $P \models \check{X} \cap \dot{G} \subseteq \{ x_\alpha : \alpha < \check{\beta} \}$.
4. But for each $p \in P$ we have $p \models \check{p} \in \dot{G}$, so for each $p \in X$ we have $p \models \check{p} \in \check{X} \cap \dot{G}$, so $p \beta + 1 \models \check{p} \beta + 1 \in \check{X} \cap \dot{G}$, a contradiction.
Theorem

Suppose that $P$ satisfies c.c.c., and $X \subseteq P$ is uncountable, then there is $p \in P$ such that $p \Vdash X \cap \dot{G}$ is uncountable

Proof.

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2. If the theorem is false, for each $p$ there is $q \leq p$ and $\alpha_q$ such that $q \Vdash X \cap \dot{G} \subseteq \{ x_\alpha : \alpha < \alpha_q \} \& X \cap \dot{G} \not\subseteq \{ x_\alpha : \alpha < \beta \}$ for $\beta < \alpha_q$

3. There cannot be uncountable many conditions which force pairwise contradictory information, so $\{ \alpha_q : q \in P \}$ is countable, so it has its supremum $\beta < \omega_1$ which satisfies $P \Vdash X \cap \dot{G} \subseteq \{ x_\alpha : \alpha < \beta \}$

4. But for each $p \in P$ we have $p \Vdash \dot{G} \subseteq \dot{X}$, so for each $p \in X$ we have $p \Vdash X \cap \dot{G}$, so $p_{\beta + 1} \Vdash X \cap \dot{G}$, a contradiction.

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3. There cannot be uncountable many conditions which force pairwise contradictory information, so $\{\alpha_q : q \in P\}$ is countable, so it has its supremum $\beta < \omega_1$ which satisfies $P \models \check{X} \cap \check{G} \subseteq \{x_\alpha : \alpha < \check{\beta}\}$
Theorem

Suppose that $P$ satisfies c.c.c., and $X \subseteq P$ is uncountable, then there is $p \in P$ such that $p \models \check{X} \cap \check{G}$ is uncountable.

Proof.

1. We may w.l.o.g. assume that $X = \{x_\alpha : \alpha < \omega_1\}$.

2. If the theorem is false, for each $p$ there is $q \leq p$ and $\alpha_q$ such that $q \models \check{X} \cap \check{G} \subseteq \{\check{x}_\alpha : \alpha < \check{\alpha}_q\}$ & $\check{X} \cap \check{G} \not\subseteq \{\check{x}_\alpha : \alpha < \check{\beta}\}$ for $\check{\beta} < \check{\alpha}_q$.

3. There cannot be uncountable many conditions which force pairwise contradictory information, so $\{\alpha_q : q \in P\}$ is countable, so it has its supremum $\beta < \omega_1$ which satisfies $P \models \check{X} \cap \check{G} \subseteq \{x_\alpha : \alpha < \check{\beta}\}$.

4. But for each $p \in P$ we have $p \models \check{p} \in \check{G}$, so for each $p \in X$ we have $p \models \check{p} \in \check{X} \cap \check{G}$, so $p_{\beta+1} \models \check{p}_{\beta+1} \in \check{X} \cap \check{G}$, a contradiction.
Theorem

(CH) There is a \( c : [\omega_1]^2 \to \{0, 1\} \) such that for each pairwise disjoint family of \( k \)-element sets (\( k \in \mathbb{N} \)) \( a_\xi = \{\alpha_1^{\xi}, \ldots, \alpha_k^{\xi}\} \) of \( \omega_1 \), for each \( M : \{1, \ldots, k\} \times \{1, \ldots, k\} \to \{0, 1\} \)

\[ \exists \xi < \eta < \omega_1 \quad \forall 1 \leq i < j \leq k \quad c(\alpha_i^{\xi}, \alpha_j^{\eta}) = M(i, j). \]

We say that \( a_\xi \) and \( a_\eta \) realize matrix \( M \) and that \( c \) realizes every matrix.
(CH) There is a \( c : [\omega_1]^2 \rightarrow \{0, 1\} \) such that for each pairwise disjoint family of \( k \)-element sets \( (k \in \mathbb{N}) \) \( a_\xi = \{\alpha_1^\xi, \ldots, \alpha_k^\xi\} \) of \( \omega_1 \), for each \( M : \{1, \ldots, k\} \times \{1, \ldots, k\} \rightarrow \{0, 1\} \)

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\]

We say that \( a_\xi \) and \( a_\eta \) realize matrix \( M \) and that \( c \) realizes every matrix.

Suppose that \( c : [\omega_1]^2 \rightarrow \{0, 1\} \) realizes every matrix. Then, for each \( k \times k \) matrix \( M_0 \) there is a c.c.c. forcing \( P \) which forces that there is an uncountable pairwise disjoint family \( \{a_\xi : \xi < \omega_1\} \) such that \( a_\xi \) and \( a_\eta \) realize matrix \( M_0 \) for every \( \xi < \eta < \omega_1 \). In paricular, \( c \) does not realize every matrix.
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Suppose that \( c : [\omega_1]^2 \to \{0, 1\} \) realizes every matrix. Then, for each \( k \times k \) matrix \( M_0 \) there is a c.c.c. forcing \( P \) which forces that there is an uncountable pairwise disjoint family \( \{a_\xi : \xi < \omega_1\} \) such that \( a_\xi \) and \( a_\eta \) realize matrix \( M_0 \) for every \( \xi < \eta < \omega_1 \). In particular, \( c \) does not realize every matrix.
Theorem

Suppose that $c : [\omega_1]^2 \rightarrow \{0, 1\}$ realizes every matrix. Then, for each $k \times k$ matrix $M_0$ there is a c.c.c. forcing $P$ which forces that there is an uncountable pairwise disjoint family $\{a_\xi : \xi < \omega_1\}$ such that $a_\xi$ and $a_\eta$ realize matrix $M_0$ for every $\xi < \eta < \omega_1$. In particular, $c$ does not realize every matrix.

Proof.

1. Fix $c : [\omega_1]^2 \rightarrow \{0, 1\}$, Suppose $c$ realizes every matrix.
Theorem

Suppose that \( c : [\omega_1]^2 \to \{0, 1\} \) realizes every matrix. Then, for each \( k \times k \) matrix \( M_0 \) there is a c.c.c. forcing \( P \) which forces that there is an uncountable pairwise disjoint family \( \{a_\xi : \xi < \omega_1\} \) such that \( a_\xi \) and \( a_\eta \) realize matrix \( M_0 \) for every \( \xi < \eta < \omega_1 \). In particular, \( c \) does not realize every matrix.

Proof.

1. Fix \( c : [\omega_1]^2 \to \{0, 1\} \), Suppose \( c \) realizes every matrix.

2. Fix a \( k \times k \) matrix \( M_0 \). Construct a forcing \( P \) consisting of all pairwise disjoint finite families \( p \) of \( k \)-element sets such that if \( a, b \in p \) and \( a < b \), then \( a \) and \( b \) realize \( M_0 \).
Theorem

Suppose that $c : [\omega_1]^2 \rightarrow \{0, 1\}$ realizes every matrix. Then, for each $k \times k$ matrix $M_0$ there is a c.c.c. forcing $P$ which forces that there is an uncountable pairwise disjoint family $\{a_\xi : \xi < \omega_1\}$ such that $a_\xi$ and $a_\eta$ realize matrix $M_0$ for every $\xi < \eta < \omega_1$. In paricular, $c$ does not realize every matrix.

Proof.

1. Fix $c : [\omega_1]^2 \rightarrow \{0, 1\}$, Suppose $c$ realizes every matrix.

2. Fix a $k \times k$ matrix $M_0$. Construct a forcing $P$ consisting of all pairwise disjoint finite families $p$ of $k$-element sets such that if $a, b \in p$ and $a < b$, then $a$ and $b$ realize $M_0$.

3. The assumption that $c$ realizes every matrix implies that $P$ is c.c.c.
Theorem

Suppose that $c : [\omega_1]^2 \to \{0, 1\}$ realizes every matrix. Then, for each $k \times k$ matrix $M_0$ there is a c.c.c. forcing $P$ which forces that there is an uncountable pairwise disjoint family $\{a_\xi : \xi < \omega_1\}$ such that $a_\xi$ and $a_\eta$ realize matrix $M_0$ for every $\xi < \eta < \omega_1$. In particular, $c$ does not realize every matrix.

Proof.

1. Fix $c : [\omega_1]^2 \to \{0, 1\}$, Suppose $c$ realizes every matrix.

2. Fix a $k \times k$ matrix $M_0$. Construct a forcing $P$ consisting of all pairwise disjoint finite families $p$ of $k$-element sets such that if $a, b \in p$ and $a < b$, then $a$ and $b$ realize $M_0$.

3. The assumption that $c$ realizes every matrix implies that $P$ is c.c.c.

4. Given any uncountable family $(p_\xi : \xi < \omega_1)$ where $p_\xi$ has just one element, by the previous discussion we have a condition $p$ which force that $G \cap \{p_\xi : \xi < \omega_1\}$ is uncountable, completing the proof.
Theorem

\((\text{MA} \vdash \neg \text{CH})\) For each \(c : [\omega_1]^2 \rightarrow \{0, 1\}\) there is \(k \in \mathbb{N}\) (arbitrary big) and there is pairwise disjoint family of \(k\)-element sets \(a_\xi = \{\alpha_1^\xi, \ldots, \alpha_k^\xi\}\) of \(\omega_1\), and there is \(M : \{1, \ldots, k\} \times \{1, \ldots, k\} \rightarrow \{0, 1\}\) such that

\[\forall \xi < \eta < \omega_1 \quad \exists 1 \leq i < j \leq k \quad c(\alpha_i^\xi, \alpha_j^\eta) \neq M(i, j).\]

i.e., no \(c\) realizes every matrix.
Example 3. Composing functions with the generic function
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Motivation:

Let $P$ consists of all functions $p : \{1, \ldots, n\} \to \omega$ with the inverse inclusion (i.e., the Cohen forcing). Let $c : \omega \to \omega$ be the generic function, i.e., $\dot{c} = \bigcup \{ p : p \in \dot{G} \}$.

Use $c$ to get the consistency of the existence of the Souslin tree, i.e., an uncountable tree without uncountable branches and without uncountable antichains.
Example 3. Composing functions with the generic function

Motivation:

Let $P$ consists of all functions $p : \{1, ..., n\} \rightarrow \omega$ with the inverse inclusion (i.e., the Cohen forcing). Let $c : \omega \rightarrow \omega$ be the generic function i.e, $\dot{c} = \bigcup \{\rho : \rho \in \dot{G}\}$. Use $c$ to get the consistency of the existence of the Souslin tree i.e., an uncountable tree without uncountable branches and without uncountable antichains.
Example 3. Composing functions with the generic function

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Definition

Let $e_\alpha : \alpha \rightarrow \omega$ for $\alpha < \omega_1$ be bijections. We say that $(e_\alpha)_{\alpha < \omega_1}$ is coherent iff

$$\forall \alpha < \beta < \omega_1 \quad \{ \xi < \alpha : e_\alpha(\xi) \neq e_\beta(\xi) \} \text{ is finite.}$$
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**Theorem**

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T((e_\alpha)_{\alpha<\omega_1}) = \{ f : \alpha \in \omega_1, f : \alpha \to \omega \ \{ \xi < \alpha : f(\xi) \neq e_\alpha(\xi) \} \text{ is finite} \}
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with inclusion forms an Aronszajn tree, i.e., without an uncountable branch.
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$P$ forces that $T = T((\dot{c} \circ \dot{e}_\alpha)_{\alpha < \omega_1})$ is a Souslin tree, i.e., it has no uncountable antichains nor uncountable branches.
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Proof.

1. Suppose \( P \models (\dot{c} \circ f_\alpha)_{\alpha < \omega_1} \) is an uncountable antichain in \( T \)
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Generic sets

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3. Since $P$ is countable may w.l.o.g. assume that $p_\alpha = p : \{1, ..., n\} \to \omega$ for all $\alpha < \omega_1$. 
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4. Take $F_\alpha = f_\alpha^{-1}[\{1, \ldots, n\}] \subseteq \omega_1$, and assume the $F_\alpha$'s for a $\Delta$-system with root $\Delta$ and that all $f_\alpha$'s agree on $\Delta$. 
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5. Choose any $f_\alpha$ and $f_\beta$ and find $m \in \omega$ such that $m \geq n$ and

$$\{\xi : f_\alpha(\xi) \neq f_\beta(\xi)\} \subseteq f_\alpha^{-1}[\{1, \ldots, m\}], f_\beta^{-1}[\{1, \ldots, m\}].$$
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6. Put $q = p \cup 0|[n + 1, m]$. Because $q \forces \dot{q} \in \dot{G}$, we have $q \forces \dot{q} \subseteq \dot{c}$, and so $q$ forces that $c \circ \dot{f}_\alpha$ and $c \circ \dot{f}_\beta$ are compatible.