Examples concerning generic sets

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Outline

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1. Generic sets in any forcing - the decision property

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2. Generic sets in c.c.c. forcings - catching uncountable sets

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3. Generics in the Cohen forcing - composing functions with the generic function

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6. If A is an atomless complete Boolean algebra then $[\phi] = \bigvee\{a \in A^* : a \Vdash \phi\}$
7. If we can prove that $P \Vdash \phi$, then we can prove that Con(ZFC) implies Con(ZFC+ϕ)

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Proof.

For each $a \in A^*$ consider the dense set in A^*

$$D_a = \{p \in A^* : p \leq a \text{ or } p \leq -a\}$$

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2. $[\neg\phi] = -[\phi]$
3. $P \Vdash [\neg\phi] \in \dot{G}$ *or* $[\phi] \in \dot{G}$

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4. If P is countable, then it is c.c.c.

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Definition

Let $B \subseteq A$ be Boolean algebras. We say that B is deep in A if and only if

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It is consistent with arbitrary big continuum that each $A \subseteq \wp(N)$ of countable independence has an ultrafilter which is countably or ω_1-generated.

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3. There cannot be uncountable many conditions which force pairwise contradictory information, so $\{\alpha_q : q \in P\}$ is countable, so it has its supremum $\beta < \omega_1$ which satisfies $P \Vdash \check{X} \cap \dot{G} \subseteq \{x_\alpha : \alpha < \check{\beta}\}$

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4. But for each $p \in P$ we have $p \Vdash \check{p} \in \dot{G}$, so for each $p \in X$ we have $p \Vdash \check{p} \in \check{X} \cap \dot{G}$, so $p_{\beta+1} \Vdash \check{p}_{\beta+1} \in \check{X} \cap \dot{G}$, a contradiction.

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Theorem

(CH) There is a $c : [\omega_1]^2 \to \{0, 1\}$ *such that for each pairwise disjoint family of* k*-element sets (*$k \in N$*)* $a_\xi = \{\alpha_1^\xi, ..., \alpha_k^\xi\}$ *of* ω_1*, for each* $M : \{1, ..., k\} \times \{1, ..., k\} \to \{0, 1\}$

$$\exists \xi < \eta < \omega_1 \qquad \forall 1 \leq i < j \leq k \qquad c(\alpha_i^\xi, \alpha_j^\eta) = M(i, j).$$

We say that a_ξ *and* a_η *realize matrix* M *and that* c *realizes every matrix.*

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Suppose that $c : [\omega_1]^2 \to \{0,1\}$ *realizes every matrix. Then, for each* $k \times k$ *matrix* M_0 *there is a c.c.c. forcing* P *which forces that there is an uncountable pairwise disjoint family* $\{a_\xi : \xi < \omega_1\}$ *such that* a_ξ *and* a_η *realize matrix* M_0 *for every* $\xi < \eta < \omega_1$*. In paricular,* c *does not realize every matrix.*

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Proof.

1. Fix $c : [\omega_1]^2 \to \{0, 1\}$, Suppose c realizes every matrix.
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3. The assumption that c realizes every matrix implies that P is c.c.c.
4. Given any uncountable family $(p_\xi : \xi < \omega_1)$ where p_ξ has just one element, by the previous discussion we have a condition p which force that $\dot{G} \cap \{p_\xi : \xi < \omega_1\}$ is uncountable, completing the proof.

□

Theorem

(MA+$\neg$CH) For each $c : [\omega_1]^2 \to \{0,1\}$ *there is* $k \in N$ *(arbitary big) and there is pairwise disjoint family of* k*-element sets* $a_\xi = \{\alpha_1^\xi, ..., \alpha_k^\xi\}$ *of* ω_1*, and there is* $M : \{1, ..., k\} \times \{1, ..., k\} \to \{0,1\}$ *such that*

$$\forall \xi < \eta < \omega_1 \qquad \exists 1 \leq i < j \leq k \qquad c(\alpha_i^\xi, \alpha_j^\eta) \neq M(i,j).$$

i.e., no c realizes every matrix.

Example 3. Composing functions with the generic function

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Let P consists of all functions $p : \{1, ..., n\} \to \omega$ with the inverse inclusion (i.e., the Cohen forcing). Let $c : \omega \to \omega$ be the generic function i.e, $\dot{c} = \bigcup\{p : p \in \dot{G}\}$.

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Use c to get the consistency of the existence of the Souslin tree i.e., an uncountable tree without uncountable branches and without uncountable antichains

Definition

Let $e_\alpha : \alpha \to \omega$ for $\alpha < \omega_1$ be bijections. We say that $(e_\alpha)_{\alpha<\omega_1}$ is coherent iff

$$\forall \alpha < \beta < \omega_1 \quad \{\xi < \alpha : e_\alpha(\xi) \neq e_\beta(\xi)\} \text{ is finite.}$$

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$$T((e_\alpha)_{\alpha<\omega_1}) = \{f : \alpha \in \omega_1,\ f : \alpha \to \omega\ \{\xi < \alpha : f(\xi) \neq e_\alpha(\xi)\} \textit{ is finite}\}$$

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4. Take $F_\alpha = f_\alpha^{-1}[\{1, ...n\}] \subseteq \omega_1$, and assume the F_αs for a Δ-system with root Δ and that all f_α's agree on Δ.
5. Choose any f_α and f_β and find $m \in \omega$ such that $m \geq n$ and

$$\{\xi : f_\alpha(\xi) \neq f_\beta(\xi)\} \subseteq f_\alpha^{-1}[\{1, ..., m\}], f_\beta^{-1}[\{1, ..., m\}].$$

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1. Suppose $P \Vdash (\dot{c} \circ \dot{f}_\alpha)_{\alpha<\omega_1}$ is an uncountable antichain in T
2. Take p_α and $f_\alpha \in T((e_\alpha)_{\alpha<\omega_1})$ such that $p_\alpha \Vdash \check{f}_\alpha = \dot{f}_\alpha$,
3. Since P is countable may w.l.o.g. assume that $p_\alpha = p : \{1, ..., n\} \to \omega$ for all $\alpha < \omega_1$.
4. Take $F_\alpha = f_\alpha^{-1}[\{1, ...n\}] \subseteq \omega_1$, and assume the F_αs for a Δ-system with root Δ and that all f_α's agree on Δ.
5. Choose any f_α and f_β and find $m \in \omega$ such that $m \geq n$ and

$$\{\xi : f_\alpha(\xi) \neq f_\beta(\xi)\} \subseteq f_\alpha^{-1}[\{1, ..., m\}], f_\beta^{-1}[\{1, ..., m\}].$$

6. Put $q = p \cup 0|[n+1, m]$. Because $q \Vdash \check{q} \in \dot{G}$, we have $q \Vdash \check{q} \subseteq \dot{c}$, and so q forces that $c \circ \dot{f}_\alpha$ and $c \circ \dot{f}_\beta$ are compatible.