BPFA and BAAFA

Their equiconsistency and their nonequivalence

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Definition (Baumgartner, 1983)

A poset $(P, \leqslant_0)$ satisfies Axiom A if and only if

- There exists a countably infinite sequence $\leqslant_0, \leqslant_1, \leqslant_2, \ldots$ of partial orders on the set P such that
- $p \leqslant_{n+1} q$ implies $p \leqslant_n q$ for all $n < \omega$ and all $p, q \in P$.
- Given $p_0 \geqslant_0 p_1 \geqslant_1 p_2 \geqslant_2 \ldots$ there exists a condition $q \in P$ such that $q \leqslant_n p_n$ for all $n < \omega$.
- Given $p \in P$, an antichain $A \subset P$ and an $n < \omega$ there exists a $q \leqslant_n p$ such that $\{r \in A | r \|_0 q\}$ is countable.

Examples

- Whatever forcing satisfies the ccc does also satisfy Axiom A.
- Proof: Let $\leqslant_n$ be the identity for all $n \in \omega \setminus 1$.
- Any countably closed notion of forcing satisfies Axiom A.
- Proof: Let $\leqslant_n$ be $\leqslant_0$ for all $n \in \omega \setminus 1$.

Definition

Let $\mathbb{P}$ be a poset and p a condition of $\mathbb{P}$. The strengthened proper game for $\mathbb{P}$ below p is played as follows:

- In move n Player I plays a $\mathbb{P}$-name for an ordinal $\dot{\alpha}_n$ ...
- ...to which Player II responds by playing a countable set of ordinals B_n.

Player II wins iff there is a $q \leqslant p$ such that $q \Vdash_{\mathbb{P}}$ "$\forall n < \omega : \dot{\alpha}_n \in \check{B}_n$".

Remark

Whenever Player II has a winning strategy in the strengthened proper game for $\mathbb{P}$ below p she has one in the proper game for $\mathbb{P}$ below p.

Definition

An ordinal α is Σ_n-correct iff $V_\alpha \prec_{\Sigma_n} V$.

Fact

The regular Σ_1-correct cardinals are precisely the inaccessible ones.

Fact

There are unboundedly many regular Σ_n-correct cardinals below any regular Σ_{n+1}-correct cardinal.

Fact

For any $n < \omega$ there are stationarily many regular Σ_n-correct cardinals below the first Mahlo cardinal.

Definition

Let κ, λ be cardinals and $\mathcal{C}$ be a class of forcing notions.
The Bounded Forcing Axiom for $\mathcal{C}$ and κ, bounded by λ—$\mathrm{BFA}(\mathcal{C}, \kappa, \lambda)$ for short—asserts the following:
If $\mathbb{P}$ is a forcing notion in $\mathcal{C}$ and $\mathcal{A}$ is a family of less than κ maximal antichains each of which has size less than λ,
there is a filter $G \subset \mathbb{P}$ such that $\forall A \in \mathcal{A} : A \cap G \supsetneq 0$.

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Examples

- The Bounded Proper Forcing Axiom BPFA is BFA$(\mathfrak{B} \cap \mathcal{P}_{\text{rop}}, \aleph_2, \aleph_2)$.
- Bounded Martins Maximum BMM is BFA$(\mathfrak{B} \cap \text{ssp}, \aleph_2, \aleph_2)$.
- Martin's Axiom for $\aleph_1$ $\text{MA}_{\aleph_1}$ (MA $+ \neg$ CH) is BFA$(\mathfrak{B} \cap \text{c.c.c.}, \aleph_2, \aleph_1)$ or BFA$(\text{c.c.c.}, \aleph_2, \Omega)$.

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Theorem (Bagaria, 2000)

If $\kappa = \lambda$ and $\mathcal{C} \subset \mathfrak{B}$ the corresponding statement is equivalent to a principle of generic absoluteness, i.e.

$$\text{BFA}(\mathcal{C}, \kappa, \kappa) \iff \text{"All } \Sigma_1\text{-statements with parameters from } H_\kappa \text{ forcable by a forcing notion from } \mathcal{C} \text{ are true."}$$

Definition (W.)

- A class of forcing notions $\mathcal{C}$ is called reasonable iff for any forcing notion $\mathbb{P} \in \mathcal{C}$, an arbitrary forcing notion $\mathbb{Q}$ and a complete Boolean algebra $\mathbb{B}$ the following holds: If there are dense embeddings $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}, \delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$, then $\mathbb{Q} \in \mathcal{C}$.
- The reasonable hull $\mathfrak{rh}(C)$ of a class of forcing notions $\mathcal{C}$ consists of all forcing notions $\mathbb{P}$ such that there exists a forcing notion $\mathbb{Q} \in \mathcal{C}$, a complete Boolean algebra $\mathbb{B}$ and dense embeddings $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$, $\delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$.
- Let $\mathcal{A}^* := \mathfrak{rh}(\mathcal{A})$ be the class of forcing notions satisfying Axiom A*.

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- Let $\mathcal{AA}^* := \mathfrak{rh}(\mathcal{AA})$ be the class of forcing notions satisfying Axiom A*.

Remark

ssp, *$\mathcal{P}_{rop}$ and $\mathcal{AA}^*$ are reasonable.*

Definition (W.)

BAAFA $:\Longleftrightarrow$ BFA$(\mathfrak{B} \cap \mathcal{AA}^*, \aleph_2, \aleph_2)$ is the Bounded Axiom A Forcing Axiom.

Theorem (Todorčević)

BAAFA $\Rightarrow \aleph_2$ *is regular and* Σ_2*-correct in* L.

Theorem (Moore, 2005)

BPFA $\Rightarrow 2^{\aleph_0} = \aleph_2$

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Lemma

Whenever $\mathbb{P}$ *is a notion of forcing satisfying Axiom* A^* *and* $p \in \mathbb{P}$*, Player II has a winning strategy in the strengthened proper game for* $\mathbb{P}$ *below* p.

Corollary

If a notion of forcing satisfies Axiom A^* *then it is proper.*

Example (Adding a club with finite conditions)

Consider the following notion of forcing:

$$\mathbb{P}_{\text{acfc}} := \{p | \overline{\overline{p}} < \aleph_0 \wedge \text{ran}(p) \subset \aleph_1 \wedge \exists f \supset p : f \text{ is a normal function.}\}$$

Lemma

$\mathbb{P}_{acfc}$ is proper.

Lemma

$\mathbb{P}_{acfc}$ does not satisfy Axiom A.*

Theorem (Shelah, 1983)

The countable support iteration of proper notions of forcing is proper.

Fact

If κ is regular and Σ_2-correct and $\mathbb{P} \in H_\kappa$ then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "κ is regular and Σ_2-correct.".

Fact

Being proper is a Σ_2-property.

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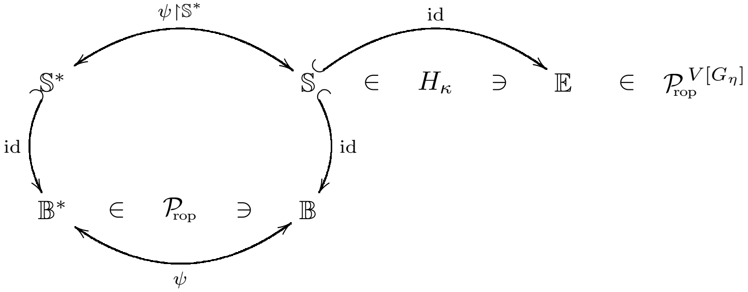
Theorem (Shelah, 1995)

If κ is regular and Σ_2-correct there is a forcing iteration $\mathbb{P}_\kappa$ which is proper and satisfies the κ-c.c. such that

$$V[G] \models \text{"ZFC} + \text{BPFA"}$$

The diagram


$\psi\restriction\mathbb{S}^*$
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H_κ
$\ni$
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$\in$
$\mathcal{P}_{\mathrm{rop}}^{V[G_\eta]}$
id
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$\mathbb{B}^*$
$\in$
$\mathcal{P}_{\mathrm{rop}}$
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$\mathbb{B}$
ψ

Theorem (Koszmider, 1993)

The countable support iteration of Axiom A forcings satisfies Axiom A.

Corollary

The countable support iteration of Axiom A^* forcings satisfies Axiom A^*.

Fact

To satisfy Axiom A^* is a Σ_2-property.

Theorem (Koszmider, 1993)

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Corollary

The countable support iteration of Axiom A forcings satisfies Axiom A*.*

Fact

To satisfy Axiom A is a Σ_2-property.*

Proof.

$\exists \mathbb{B}, Q, X, \langle \leqslant^n | n < \omega\rangle, \delta_{\mathbb{P}}, \delta_{\mathbb{Q}}, f \Big(\mathbb{B}$ is a complete Boolean algebra, $\delta_{\mathbb{P}}$ is a dense embedding of $\mathbb{P}$ into $\mathbb{B}, \forall S \subset Q : S \in X, \mathrm{dom}(f) = Q \times X \times \omega \times \omega, \forall n < \omega \big(\leqslant^n$ is a partial ordering of Q and $\forall p, q \in Q(p \leqslant^{n+1} q \to p \leqslant^n q)\big), \delta_{\mathbb{Q}}$ is a dense embedding of $(Q, \leqslant^0)$ into $\mathbb{B}, \forall \langle q_n | n < \omega\rangle \big((\forall n < \omega : q_{n+1} \leqslant^n q_n) \to \exists q \in Q \forall n < \omega : q \leqslant^n q_n \big)$ and $\forall q \in Q \forall n < \omega \forall A \in X (A$ is an antichain $\to \exists r \in Q(r \leqslant^n q \wedge \{s \in A | s ||^0 r\} \subset f``(\{q\} \times \{A\} \times \{n\} \times \omega)) \Big)$

⊣

Theorem (W., 2007)

If κ is regular and Σ_2-correct then there is a forcing $\mathbb{Q}_\kappa$ satisfying both Axiom A^* and the κ-c.c. such that in the generic extension we have

$$\mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \mathrm{BAAFA} + \neg\,\mathrm{BPFA}$$

First part of the proof.

One defines a forcing iteration analogous to the one above. Simply substitute "Axiom A*" for "proper" throughout the whole construction. This shows that

$$V[G] \models \mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \mathrm{BAAFA}$$

⊣

Lemma

$p \in \mathbb{P}_{acfc}$ is a $\Delta_1(\{\aleph_1, p\})$-relation.

Proof.

The original defintion yields the $\Sigma_1(\{\aleph_1\})$-definition:

$$\exists f \supset p\big(f \in \text{Func} \wedge \forall \alpha, \beta \in \text{dom}(f)(\alpha < \beta \rightarrow f(\alpha) < f(\beta)) \\ \wedge \forall \alpha \in \text{Lim} \cap \text{dom}(f), \beta < f(\alpha) \exists \gamma < \alpha : f(\gamma) > \beta\big).$$

A $\Pi_1(\{\aleph_1\})$-defintion is provided by the following formula:

$$p \in \text{Func} \wedge \text{dom}(p) \subset \aleph_1 \wedge \nexists g : \omega \hookrightarrow \text{dom}(p) \wedge \forall g, \langle g_\gamma | \gamma < \beta \rangle, \alpha \in \text{dom}(p) \\ \big((\alpha < \beta \wedge \beta \in \text{dom}(p) \wedge \nexists \gamma \in \text{dom}(p) : \alpha < \gamma \wedge \gamma < \beta) \rightarrow \big(p(\alpha) < p(\beta) \wedge \\ \beta \leqslant p(\beta) \wedge \nexists \gamma < \beta (g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha \text{ is order-preserving.}) \wedge (\beta \in \text{Lim} \\ \rightarrow \forall \gamma < p(\beta) \exists \eta < \beta \nexists \zeta < \beta (g_\eta : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta \text{ is order-preserving.}))\big)\big)$$

⊣

Corollary

$\mathbb{P}_{acfc}$ is identical in any two transitive models of set theory which share their $\aleph_1$.

Lemma

Let $\langle \alpha_n | n < \omega \rangle$ be a sequence of countable indecomposable ordinals and $\langle \beta_n | n < \omega \rangle$ a sequence of ordinals such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. The following sets are dense in $\mathbb{P}_{acfc}$:

$$D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle} := \{p \in \mathbb{P}_{acfc} | \exists n < \omega, \gamma \in \aleph_1 \setminus \beta_n : (\alpha_n, \gamma) \in q\}$$

Proposition

$\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}_\kappa}$ "$\neg$ BFA $\left(\mathrm{ro}(\mathbb{P}_{acfc}), \aleph_2, \aleph_2\right)$".

Proof.

Suppose $q \Vdash_{\mathbb{Q}_\kappa}$ "$\text{BFA}\left(\text{ro}(\mathbb{P}_{acfc}), \aleph_2, \aleph_2\right)$".

$$\mathcal{D} := \{D | D \subset \mathbb{P}_{acfc} \wedge D \textit{ is dense.}\}$$

Proof.

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Let $G \ni q$ *be* $\mathbb{Q}_\kappa$*-generic and* $\mathbb{B} := \mathrm{ro}^{V[G]}(\mathbb{P}_{acfc})$, $\delta : \mathbb{P}_{acfc} \longrightarrow \mathbb{B}$ *the corresponding dense embedding and* $\mathcal{D}_{\mathbb{B}} := \{\delta``D | D \in \mathcal{D}\}$.

Proof.

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$\mathbb{B}$ *is proper. So*

$$q \Vdash_{\mathbb{Q}_\kappa} \text{“}\exists H : H \text{ is a } \check{\mathcal{D}}_{\mathbb{B}}\text{-generic filter over } \mathbb{B}.\text{”}$$

Define a normal function in $V[G]$*:*

$$f : \aleph_1 \longrightarrow \aleph_1$$
$$\alpha \longmapsto \text{the } \beta < \aleph_1 \text{ such that } \exists p \in \mathbb{P}_{acfc}\left(\alpha \in \text{dom}(p) \wedge p(\alpha) = \beta \wedge \delta(p) \in H\right)$$

- *In move* 0 *play* 0 *(or any other ordinal name).*
- *In move* n *our opponent plays a* $B_n \in [\Omega]^{<\omega_1}$.
- *In move* $n+1$ *we choose an indecomposable countable ordinal* α_{n+1} *greater than* $\beta_n := \sup B_n + 1$ *and play* $\dot{f}(\alpha_{n+1})$.

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This yields a sequence of indecomposable ordinals $(\alpha_n | n < \omega)$ *and a sequence of ordinals* $(\beta_n | n < \omega)$ *such that* $\forall n < \omega : \beta_n < \alpha_{n+1}$*.* $D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$ *is dense and in* V*, since our game was played there. Let* Λ *be a name for* H*, then*

$$q \Vdash_{\mathbb{Q}_\kappa} \text{“}\Lambda \cap \dot{\delta}\text{“}D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle} \supsetneq \emptyset\text{”}. \tag{1}$$

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Let $s \leqslant_{\mathbb{Q}_\kappa} q$ be arbitrarily chosen. By (1) there is a $p \in D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$ and $u \leqslant_{\mathbb{Q}_\kappa} s$ such that $u \Vdash_{\mathbb{Q}_\kappa}$ “$\dot{\delta}(\check{p}) \in \Lambda$”.
By defintion of $D^{\langle \alpha_n | n<\omega \rangle}_{\langle \beta_n | n<\omega \rangle}$ there are $n < \omega$, $\gamma \in \aleph_1 \setminus \beta_n$ such that $(\alpha_n, \gamma) \in p$.
But then $u \Vdash_{\mathbb{Q}_\kappa}$ “$\dot{f}(\alpha_n) = \check{\gamma}$” hence $u \Vdash_{\mathbb{Q}_\kappa}$ “$\dot{f}(\alpha_n) \notin \check{B}_n$”.

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Does BAAFA decide the size of the continuum?