BPFA and BAAFA

Their equiconsistency and their nonequivalence

Thilo Weinert

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1. Basics
   - Axiom A
   - The strengthened proper game
   - $\Sigma_n$-correct cardinals

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**Definition (Baumgartner, 1983)**

A poset \((P, \leq_0)\) satisfies **Axiom A** if and only if

- There exists a countably infinite sequence \(\leq_0, \leq_1, \leq_2, \ldots\) of partial orders on the set \(P\) such that
- \(p \leq_{n+1} q\) implies \(p \leq_n q\) for all \(n < \omega\) and all \(p, q \in P\).
- Given \(p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots\) there exists a condition \(q \in P\) such that \(q \leq_n p_n\) for all \(n < \omega\).
- Given \(p \in P\), an antichain \(A \subset P\) and an \(n < \omega\) there exists a \(q \leq_n p\) such that \(\{r \in A \mid r \parallel_0 q\}\) is countable.

**Examples**

- Whatever forcing satisfies the ccc does also satisfy Axiom A.
  
  Proof: Let \(\leq_n\) be the identity for all \(n \in \omega \setminus 1\).

- Any countably closed notion of forcing satisfies Axiom A.
  
  Proof: Let \(\leq_n\) be \(\leq_0\) for all \(n \in \omega \setminus 1\).
The strengthened proper game

Definition

Let $\mathbb{P}$ be a poset and $p$ a condition of $\mathbb{P}$. The **strengthened proper game** for $\mathbb{P}$ below $p$ is played as follows:

- In move $n$ Player I plays a $\mathbb{P}$-name for an ordinal $\dot{\alpha}_n$...
- ... to which Player II responds by playing a countable set of ordinals $B_n$.

Player II wins iff there is a $q \leq p$ such that $q \Vdash_{\mathbb{P}} \forall n < \omega : \dot{\alpha}_n \in \dot{B}_n$.

Remark

*Whenever Player II has a winning strategy in the strengthened proper game for $\mathbb{P}$ below $p$ she has one in the proper game for $\mathbb{P}$ below $p$.*
**Definition**

An ordinal $\alpha$ is $\Sigma_n$-correct iff $V_\alpha \prec_\Sigma V$.

**Fact**

The regular $\Sigma_1$-correct cardinals are precisely the inaccessible ones.

**Fact**

There are unboundedly many regular $\Sigma_n$-correct cardinals below any regular $\Sigma_{n+1}$-correct cardinal.

**Fact**

For any $n < \omega$ there are stationarily many regular $\Sigma_n$-correct cardinals below the first Mahlo cardinal.
Definition

Let $\kappa, \lambda$ be cardinals and $C$ be a class of forcing notions. The **Bounded Forcing Axiom** for $C$ and $\kappa$, bounded by $\lambda$—BFA($C, \kappa, \lambda$) for short—asserts the following:

If $\mathbb{P}$ is a forcing notion in $C$ and $A$ is a family of less than $\kappa$ maximal antichains each of which has size less than $\lambda$,
there is a filter $G \subset \mathbb{P}$ such that $\forall A \in A : A \cap G \supseteq 0$. 
**Definition**

Let $\kappa, \lambda$ be cardinals and $\mathcal{C}$ be a class of forcing notions. The **Bounded Forcing Axiom** for $\mathcal{C}$ and $\kappa$, bounded by $\lambda$—$\text{BFA}(\mathcal{C}, \kappa, \lambda)$ for short—asserts the following:

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**Examples**

- The Bounded Proper Forcing Axiom BPFA is $\text{BFA}(\mathfrak{B} \cap \mathcal{P}_{\text{top}}, \aleph_2, \aleph_2)$.
- Bounded Martins Maximum BMM is $\text{BFA}(\mathfrak{B} \cap \mathcal{S}_{\text{sp}}, \aleph_2, \aleph_2)$.
- Martin’s Axiom for $\aleph_1$ MA$_{\aleph_1}$ (MA $\not\rightarrow$ CH) is $\text{BFA}(\mathfrak{B} \cap \text{c.c.c.}, \aleph_2, \aleph_1)$ or $\text{BFA}(\text{c.c.c.}, \aleph_2, \Omega)$.
Definition

Let $\kappa, \lambda$ be cardinals and $C$ be a class of forcing notions. The **Bounded Forcing Axiom** for $C$ and $\kappa$, bounded by $\lambda$—$\text{BFA}(C, \kappa, \lambda)$ for short—asserts the following:

If $P$ is a forcing notion in $C$ and $A$ is a family of less than $\kappa$ maximal antichains each of which has size less than $\lambda$, there is a filter $G \subseteq P$ such that $\forall A \in A : A \cap G \not\supseteq 0$.

Examples

- The Bounded Proper Forcing Axiom BPFA is $\text{BFA}(\mathcal{B} \cap \mathcal{P}_{\text{top}}, \aleph_2, \aleph_2)$.
- Bounded Martins Maximum BMM is $\text{BFA}(\mathcal{B} \cap \mathcal{ssp}, \aleph_2, \aleph_2)$.
- Martin’s Axiom for $\aleph_1 \text{ MA}_{\aleph_1}$ $(\text{MA} + \neg \text{CH})$ is $\text{BFA}(\mathcal{B} \cap \text{c.c.c.}, \aleph_2, \aleph_1)$ or $\text{BFA}(\text{c.c.c.}, \aleph_2, \Omega)$.

Theorem (Bagaria, 2000)

*If $\kappa = \lambda$ and $C \subseteq \mathcal{B}$ the corresponding statement is equivalent to a principle of generic absoluteness, i.e.*

$$\text{BFA}(C, \kappa, \kappa) \iff \text{“All } \Sigma_1 \text{-statements with parameters from } H_\kappa \text{ forcable by a forcing notion from } C \text{ are true.”}$$
**Definition (W.)**

- A class of forcing notions $C$ is called **reasonable** iff for any forcing notion $P \in C$, an arbitrary forcing notion $Q$ and a complete Boolean algebra $B$ the following holds: If there are dense embeddings $\delta_P : P \rightarrow B, \delta_Q : Q \rightarrow B$, then $Q \in C$.

- The **reasonable hull** $\mathfrak{h}(C)$ of a class of forcing notions $C$ consists of all forcing notions $P$ such that there exists a forcing notion $Q \in C$, a complete Boolean algebra $B$ and dense embeddings $\delta_P : P \rightarrow B, \delta_Q : Q \rightarrow B$.

- Let $\mathcal{A}^* := \mathfrak{h}(\mathcal{A})$ be the class of forcing notions satisfying Axiom A*.
**Definition (W.)**

- A class of forcing notions \( C \) is called **reasonable** iff for any forcing notion \( P \in C \), an arbitrary forcing notion \( Q \) and a complete Boolean algebra \( B \) the following holds: If there are dense embeddings \( \delta_P : P \rightarrow B, \delta_Q : Q \rightarrow B \), then \( Q \in C \).

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- Let \( \mathcal{M}^* := \mathcal{H}(\mathcal{M}) \) be the class of forcing notions satisfying **Axiom A***.

**Remark**

\( \text{ssp}, \mathcal{P}_{\text{top}} \) and \( \mathcal{M}^* \) are reasonable.

**Definition (W.)**

\( \text{BAAFA} \iff \text{BFA}(\mathcal{B} \cap \mathcal{M}^*, \aleph_2, \aleph_2) \) is the **Bounded Axiom A Forcing Axiom**.
Some facts

**Theorem (Todorčević)**

\[
\text{BAAFA} \Rightarrow \aleph_2 \text{ is regular and } \Sigma_2\text{-correct in } L.
\]

**Theorem (Moore, 2005)**

\[
\text{BPFA} \Rightarrow 2^{\aleph_0} = \aleph_2
\]
Some facts

Theorem (Todorčević)

$\text{BAFA} \Rightarrow \aleph_2$ is regular and $\Sigma_2$-correct in $L$.

Theorem (Moore, 2005)

$\text{BPFA} \Rightarrow 2^{\aleph_0} = \aleph_2$

Lemma

Whenever $P$ is a notion of forcing satisfying Axiom $A^*$ and $p \in P$, Player II has a winning strategy in the strengthened proper game for $P$ below $p$.

Corollary

If a notion of forcing satisfies Axiom $A^*$ then it is proper.
Example (Adding a club with finite conditions)

Consider the following notion of forcing:

\[ P_{acfc} := \{ p \mid \overline{p} < \aleph_0 \wedge \text{ran}(p) \subset \aleph_1 \wedge \exists f \supset p : f \text{ is a normal function.} \} \]

Lemma

\( P_{acfc} \) is proper.

Lemma

\( P_{acfc} \) does not satisfy Axiom A*. 
Theorem (Shelah, 1983)

The countable support iteration of proper notions of forcing is proper.

Fact

If $\kappa$ is regular and $\Sigma_2$-correct and $P \in H_\kappa$ then $\mathbb{1}_P \Vdash_P "\kappa$ is regular and $\Sigma_2$-correct."

Fact

Being proper is a $\Sigma_2$-property.
Theorem (Shelah, 1983)

*The countable support iteration of proper notions of forcing is proper.*

Fact

*If $\kappa$ is regular and $\Sigma_2$-correct and $P \in H_\kappa$ then $1_P \Vdash_P "\kappa$ is regular and $\Sigma_2$-correct.".*

Fact

*Being proper is a $\Sigma_2$-property.*

Theorem (Shelah, 1995)

*If $\kappa$ is regular and $\Sigma_2$-correct there is a forcing iteration $P_\kappa$ which is proper and satisfies the $\kappa$-c.c. such that*

$$V[G] \models "\text{ZFC + BPFA}"$$
The diagram

\[
\begin{array}{c}
S^* \xrightarrow{\psi | S^*} S \xleftarrow{id} B^* \\
\xleftarrow{id} & \xrightarrow{id} & \xrightarrow{\psi} \\
B^* \xrightarrow{\in \mathcal{P}_{rop}} \exists E \in \mathcal{P}_{rop} V[G_\eta] \\
\end{array}
\]
Theorem (Koszmider, 1993)

The countable support iteration of Axiom A forcings satisfies Axiom A.

Corollary


Fact

To satisfy Axiom $A^*$ is a $\Sigma_2$-property.
Theorem (Koszmider, 1993)

The countable support iteration of Axiom A forcings satisfies Axiom A.

Corollary

The countable support iteration of Axiom A* forcings satisfies Axiom A*.

Fact

To satisfy Axiom A* is a $\Sigma_2$-property.

Proof.

$\exists \mathbb{B}, Q, X, \langle \leq_n \mid n < \omega \rangle, \delta_P, \delta_Q, f (\mathbb{B} \text{ is a complete Boolean algebra, } \delta_P \text{ is a dense embedding of } \mathbb{P} \text{ into } \mathbb{B}, \forall S \subset Q : S \in X, \text{dom}(f) = Q \times X \times \omega \times \omega, \forall n < \omega (\leq_n \text{ is a partial ordering of } Q \text{ and } \forall p, q \in Q (p \leq_{n+1} q \rightarrow p \leq_n q)), \delta_Q \text{ is a dense embedding of } (Q, \leq^0) \text{ into } \mathbb{B}, \forall \langle q_n \mid n < \omega \rangle (\forall n < \omega : q_{n+1} \leq_n q_n) \rightarrow \exists q \in Q \forall n < \omega : q \leq_n q_n \text{ and } \forall q \in Q \forall n < \omega \forall A \in X (A \text{ is an antichain } \rightarrow \exists r \in Q (r \leq_n q \land \{ s \in A \mid s \parallel^0 r \} \subset f^\omega (\{ q \} \times \{ A \} \times \{ n \} \times \omega)))$
**Theorem (W., 2007)**

*If* $\kappa$ *is regular and* $\Sigma_2$-*correct then there is a forcing* $Q_\kappa$ *satisfying both Axiom A* $^*$ *and the* $\kappa$-*c.c. such that in the generic extension we have*

$$\text{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \text{BAAFA} \vdash \neg \text{BPFA}$$

**First part of the proof.**

One defines a forcing iteration analogous to the one above. Simply substitute “Axiom A*” for “proper” throughout the whole construction. This shows that

$$V[G] \models \text{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \text{BAAFA}$$
Lemma

$p \in \mathbb{P}_{acfc} \text{ is a } \Delta_1(\{\aleph_1, p\})$-relation.

Proof.

The original definition yields the $\Sigma_1(\{\aleph_1\})$-definition:

$$\exists f \supset p \left( f \in \text{Func} \land \forall \alpha, \beta \in \text{dom}(f) (\alpha \prec \beta \rightarrow f(\alpha) < f(\beta)) \land \forall \alpha \in \text{Lim} \cap \text{dom}(f), \beta < f(\alpha) \exists \gamma < \alpha : f(\gamma) > \beta \right).$$

A $\Pi_1(\{\aleph_1\})$-definition is provided by the following formula:

$$p \in \text{Func} \land \text{dom}(p) \subset \aleph_1 \land \forall g : \omega \rightarrow \text{dom}(p) \land \forall \gamma \in \text{dom}(p) : \gamma < \beta \rightarrow (p(\alpha) < p(\beta) \land \beta \notin p(\beta) \land \forall \gamma < \beta (g : p(\beta) \setminus p(\alpha) \rightarrow \gamma \setminus \alpha \text{ is order-preserving.}) \land (\beta \in \text{Lim} \rightarrow \forall \gamma < p(\beta) \exists \eta < \beta \exists \zeta < \beta (g_\eta : p(\beta) \setminus \gamma \rightarrow \zeta \setminus \eta \text{ is order-preserving.})))$$

Corollary

$\mathbb{P}_{acfc}$ is identical in any two transitive models of set theory which share their $\aleph_1$. 
Lemma

Let $\langle \alpha_n | n < \omega \rangle$ be a sequence of countable indecomposable ordinals and $\langle \beta_n | n < \omega \rangle$ a sequence of ordinals such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. The following sets are dense in $\mathbb{P}_{acfc}$:

$$D^{\langle \alpha_n | n < \omega \rangle}_{\langle \beta_n | n < \omega \rangle} := \{ p \in \mathbb{P}_{acfc} | \exists n < \omega, \gamma \in \aleph_1 \setminus \beta_n : (\alpha_n, \gamma) \in q \}$$

Proposition

$\mathbb{I}_Q \models Q_\kappa \ “\neg \text{BFA} (\text{ro} (\mathbb{P}_{acfc}), \aleph_2, \aleph_2) \”$. 

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BPFA and BAAFA

An open question
Proof.

Suppose $q \models_{Q\kappa} \text{“BFA (} ro(P_{acf}), \aleph_2, \aleph_2 \text{”)”}.$

$$D := \{D \mid D \subset P_{acf} \land D \text{ is dense.}\}$$
Why BPFA fails

Proof.

Suppose \( q \vDash_{Q_\kappa} \text{"BFA (ro}(P_{acf}) , \aleph_2, \aleph_2)"").

\[ D := \{ D \mid D \subseteq P_{acf} \land D \text{ is dense} \} \]

Let \( G \ni q \) be \( Q_\kappa \)-generic and \( B := \text{ro}^V[G](P_{acf}), \delta : P_{acf} \rightarrow B \) the corresponding dense embedding and \( D_B := \{ \delta "D \mid D \in D \} \).
**Proof.**

Suppose $q \models_{Q_\kappa} \text{“BFA} \left( \text{ro}(\mathbb{P}_{\text{acfc}}), \aleph_2, \aleph_2 \right) \text{“}.$

\[
\mathcal{D} := \{ D \mid D \subset \mathbb{P}_{\text{acfc}} \land D \text{ is dense.} \}
\]

Let $G \ni q$ be $Q_\kappa$-generic and $\mathbb{B} := \text{ro}^V[G](\mathbb{P}_{\text{acfc}})$, $\delta : \mathbb{P}_{\text{acfc}} \rightarrow \mathbb{B}$ the corresponding dense embedding and $\mathcal{D}_{\mathbb{B}} := \{ \delta \mid D \in \mathcal{D} \}$.

$\mathbb{B}$ is proper. So

\[
q \models_{Q_\kappa} \text{“}\exists H : H \text{ is a } \mathcal{D}_{\mathbb{B}}\text{-generic filter over } \mathbb{B}. \text{“}
\]

Define a normal function in $V[G]$:

\[
f : \aleph_1 \rightarrow \aleph_1
\]

\[
\alpha \mapsto \beta < \aleph_1 \text{ such that } \exists p \in \mathbb{P}_{\text{acfc}} (\alpha \in \text{dom}(p) \land p(\alpha) = \beta \land \delta(p) \in H)
\]
Why BPFA fails

- In move 0 play 0 (or any other ordinal name).
- In move $n$ our opponent plays a $B_n \in [\Omega]^{<\omega_1}$.
- In move $n+1$ we choose an indecomposable countable ordinal $\alpha_{n+1}$ greater than $\beta_n := \sup B_n + 1$ and play $\dot{f}(\alpha_{n+1})$. 
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This yields a sequence of indecomposable ordinals $(\alpha_n|n<\omega)$ and a sequence of ordinals $(\beta_n|n<\omega)$ such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. $D^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)}$ is dense and in $V$, since our game was played there. Let $\Lambda$ be a name for $H$, then

$$q \models_{Q_\kappa} \text{“}\Lambda \cap \dot{\delta}^{D^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)}} \not\supset \emptyset\text{“}.$$  (1)
Why BPFA fails

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This yields a sequence of indecomposable ordinals $(\alpha_n|n < \omega)$ and a sequence of ordinals $(\beta_n|n < \omega)$ such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. $D^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)}$ is dense and in $V$, since our game was played there. Let $\Lambda$ be a name for $H$, then

$$q \models_{Q_\kappa} \Leftrightarrow \delta \begin{array}{|c|}
\Lambda \cap \check{D}^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)} \not\subseteq \emptyset
\end{array}.$$ (1)

Let $s \leq_{Q_\kappa} q$ be arbitrarily chosen. By (1) there is a $p \in D^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)}$ and $u \leq_{Q_\kappa} s$ such that $u \models_{Q_\kappa} \check{\delta}(\check{p}) \in \Lambda$.

By definition of $D^{(\alpha_n|n<\omega)}_{(\beta_n|n<\omega)}$ there are $n < \omega, \gamma \in \aleph_1 \setminus \beta_n$ such that $(\alpha_n, \gamma) \in p$.

But then $u \models_{Q_\kappa} \check{f}(\alpha_n) = \check{\gamma}$ hence $u \models_{Q_\kappa} \check{f}(\alpha_n) \notin \check{B}_n$.
Why BPFA fails

- In move 0 play 0 (or any other ordinal name).
- In move $n$ our opponent plays a $B_n \in [\Omega]^{<\omega_1}$.
- In move $n + 1$ we choose an indecomposable countable ordinal $\alpha_{n+1}$ greater than $\beta_n := \text{sup } B_n + 1$ and play $\dot{f}(\alpha_{n+1})$.

This yields a sequence of indecomposable ordinals $(\alpha_n | n < \omega)$ and a sequence of ordinals $(\beta_n | n < \omega)$ such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. $D^{\langle \alpha_n | n < \omega \rangle}_{\langle \beta_n | n < \omega \rangle}$ is dense and in $V$, since our game was played there. Let $\Lambda$ be a name for $H$, then

$$q \models_{Q_\kappa} \langle \Lambda \cap \delta \rangle \models D^{\langle \alpha_n | n < \omega \rangle}_{\langle \beta_n | n < \omega \rangle} \supseteq \emptyset. \quad (1)$$

Let $s \leq_{Q_\kappa} q$ be arbitrarily chosen. By (1) there is a $p \in D^{\langle \alpha_n | n < \omega \rangle}_{\langle \beta_n | n < \omega \rangle}$ and $u \leq_{Q_\kappa} s$ such that $u \models_{Q_\kappa} \langle \delta \rangle \models (\bar{p}) \in \Lambda$.

By definition of $D^{\langle \alpha_n | n < \omega \rangle}_{\langle \beta_n | n < \omega \rangle}$ there are $n < \omega$, $\gamma \in \mathbb{N}_1 \setminus \beta_n$ such that $(\alpha_n, \gamma) \in p$.

But then $u \models_{Q_\kappa} \langle \dot{f}(\alpha_n) = \dot{\gamma} \rangle$ hence $u \models_{Q_\kappa} \langle \dot{f}(\alpha_n) \notin \bar{B}_n \rangle$. 

\[ \therefore \]
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BPFA and BAFA
Does BAAFA decide the size of the continuum?